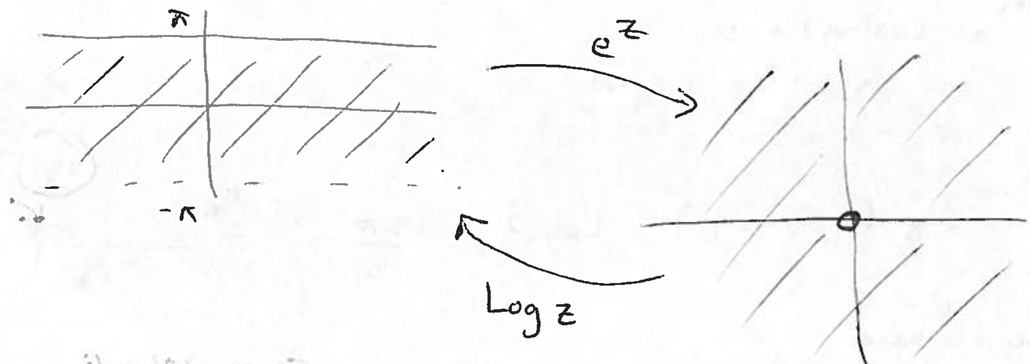
Review # 2 - Solutions

1. (a) Give the def$^{\underline{n}}$ of the Principal branch Log(z)

$$\mathrm{Log}(z) = \mathrm{Log}|z| + i\,\mathrm{Arg}\, z \quad \text{where } -\pi < \mathrm{Arg}\, z \le \pi$$

(b) domain Log z: $\mathbb{C} - \{0\}$

range $\mathrm{Log}\, z = \{ z \in \mathbb{C} \mid x \in \mathbb{R} \quad -\pi < y \le \pi \}$



(c) Is $\mathrm{Log}(z^2) = 2\mathrm{Log}(z) \quad \forall z \in \mathbb{C}$

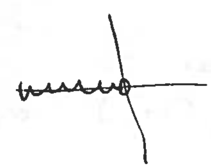
FALSE. $\mathrm{Im}(\mathrm{Log}(w))$ is always in $(-\pi, \pi]$

if $z = i$ $\quad \mathrm{Log}\, z = i\pi$

$2\mathrm{Log}\, z = 2i\pi$ (not in the right range)

in fact $\mathrm{Log}\, i^2 = \mathrm{Log}\, 1 = 0 \neq 2\pi i$

(d) Log w is analytic in $\mathbb{C} \setminus (-\infty, 0]$

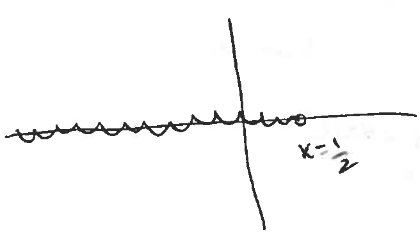


if $w = 2z - 1 = 2x + 2iy - 1$
$= 2x - 1 + 2iy$

the branch cut corresponds to $\mathrm{Re}(w) \le 0 \iff 2x - 1 \le 0$

$2x \le 1$

and $\mathrm{Im}(w) = 0 \Rightarrow y = 0 \qquad x \le \frac{1}{2}$



so $\mathrm{Log}(2z-1)$ is analytic in

$\mathbb{C} \setminus (-\infty, \frac{1}{2}]$

#2 (a) $\sinh(1+\pi i)$

we use the fact that $\sinh z = \dfrac{e^{z}-e^{-z}}{2}$ with $z = 1+\pi i$

$$\sinh(1+\pi i) = \frac{e^{1}e^{\pi i} - e^{-1}e^{-\pi i}}{2} = -\frac{e^{1}-e^{-1}}{2} = -\sinh 1$$

this is a pure real number.

$$e^{\pi i} = \cos\pi + i\sin\pi = -1$$

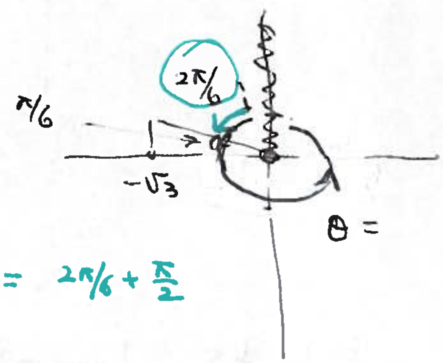
$$e^{-\pi i} = \cos(-\pi) + i\sin(-\pi) = \cos(\pi) - i\sin(\pi) = -1$$

(b) $\mathcal{L}_{\frac{\pi}{2}}(-\sqrt{3}+i) = \text{Log } 2 + i\dfrac{5\pi}{6}$

log where

$\dfrac{\pi}{2} < \arg_{\frac{\pi}{2}} \le \dfrac{5\pi}{2}$.

$|-\sqrt{3}+i| = \sqrt{3+1} = 2$



$\dfrac{5\pi}{6} = 2\pi/6 + \dfrac{\pi}{2}$

#3. $\text{Log}(z^2+1) = \dfrac{i\pi}{2}$

Take exp of both sides $\Longleftrightarrow$ $z^2+1 = e^{i\pi/2}$

but $e^{i\pi/2} = \overset{0}{\cancel{\cos\frac{\pi}{2}}} + i\sin\frac{\pi}{2} = i$ $\Longrightarrow$ $z^2+1 = i$

$z^2 = 1-i$

Consider

$\log(1+i) = \text{Log}|1+i| + i\,\text{Arg}(1+i) + 2\pi k i$ $\Longleftarrow$ $z = (1+i)^{1/2}$

$= \text{Log}\sqrt{2} + i\dfrac{3\pi}{4} + 2k\pi i$ $\qquad = e^{\frac{1}{2}\log(1-i)}$

so $z = e^{\frac{1}{2}\text{Log}\sqrt{2} + i\frac{3\pi}{8} + k\pi i}$ $\quad k = 0, \pm1, \pm2, \ldots$

$= 2^{1/4} e^{i3\pi/8 + k\pi i}$

Since $(1+i)^{1/2}$ ← there are only 2 distinct values.

$k = 0, 1$

$z_0 = 2^{1/4} e^{i3\pi/8}$ $\qquad z_1 = 2^{1/4} e^{i11\pi/8} = 2^{1/4} e^{-5\pi i/8}$

#4 Find all complex values of z for which $e^{z^2} = 1$.

Take log of both sides

$$\log(e^{z^2}=1) \Leftrightarrow \quad z^2 = \log 1 = \overset{0}{\cancel{\mathrm{Log}}\, 1} + i\, \overset{0}{\cancel{\mathrm{Arg}}\, 1} + i2k\pi \qquad k \in \mathbb{Z}$$

$$z^2 = 2k\pi i$$

$$z = \sqrt{2k\pi i} \qquad k \in \mathbb{Z}$$

$$= \sqrt{2k\pi}\,\sqrt{i}$$

then solve for $\sqrt{i} = i^{1/2}$

$$= (e^{i\pi/2})^{1/2} = 1^{1/2} e^{i\pi/4} = e^{i2k\pi/2} e^{i\pi/4} \qquad k = 0, 1$$

$$k=0 \Rightarrow e^{i\pi/4}$$

$$k=1 \Rightarrow e^{i\pi} e^{i\pi/4} = e^{i5\pi/4}$$

so $\quad z = \sqrt{2k\pi}\, e^{i\pi/4}, \; \sqrt{2k\pi}\, e^{i5\pi/4} \quad$ for $k \in \mathbb{Z}$

there are ∞ many solutions

#5 (a) $i^{\sqrt{2}} = e^{\sqrt{2}\log i}$

$$\log i = \overset{0}{\cancel{\mathrm{Log}}\,|i|} + i\,\mathrm{Arg}\, i + 2k\pi i \qquad k = 0, 1, \ldots$$

$$= \frac{i\pi}{2} + 2k\pi i$$

so $\quad i^{\sqrt{2}} = e^{\sqrt{2}(i\pi/2 + 2k\pi i)} = e^{i\sqrt{2}(\pi/2 + 2k\pi)} \qquad k \in \mathbb{Z}$

(b) $$\left(\frac{2i}{i+1}\right)^{1/3} = (1+i)^{1/3} = e^{1/3 \log(1+i)} = 2^{1/6} e^{i(\frac{\pi}{4} + 2k\pi)\frac{1}{3}} = 2^{1/6} e^{i\left(\frac{\pi + 8k\pi}{12}\right)}$$

$k = 0, 1, 2$

$$\frac{2i}{i+1} = \frac{2i}{i+1} \times \frac{1-i}{1-i} = \frac{2(i+1)}{2} = 1+i$$

$$\log(1+i) = \mathrm{Log}\,|1+i| + i\,\mathrm{Arg}(1+i) + 2k\pi i \qquad k \in \mathbb{Z}$$

$$= \mathrm{Log}\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)$$

we have 3 values

$$z_0 = 2^{1/6} e^{i\pi/12}$$

$$z_1 = 2^{1/6} e^{i9\pi/12} = 2^{1/6} e^{i3\pi/4}$$

$$z_2 = 2^{1/6} e^{i17\pi/12} = 2^{1/6} e^{-i7\pi/12}$$

5(c) $\quad (\sqrt{3}+i)^{1+i} = e^{(1+i)\log(\sqrt{3}+i)} = e^{\log(\sqrt{3}+i)}\, e^{i\log(\sqrt{3}+i)}$

$$\log(\sqrt{3}+i) = \mathrm{Log}\,|\sqrt{3}+i| + i\,\mathrm{Arg}(\sqrt{3}+i) + 2k\pi i \qquad k\in\mathbb{Z}$$
$$= \mathrm{Log}\,2 + i\frac{\pi}{6} + 2k\pi i \qquad k\in\mathbb{Z}$$

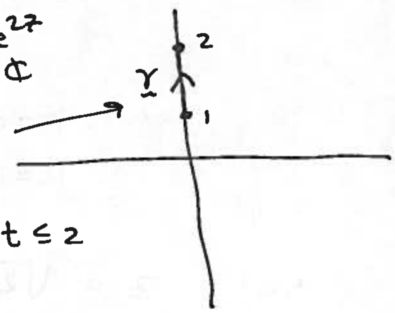
so $\quad (\sqrt{3}+i)^{1+i} = (\sqrt{3}+i)\, e^{i\log 2 \cdot i(\frac{\pi}{6}+2k\pi))}$
$$= (\sqrt{3}+i)\, e^{i\log 2}\, e^{-(\frac{\pi}{6}+2k\pi)} \qquad k\in\mathbb{Z}$$

6. Compute by two different methods.

$$\int_i^{2i} z^2 - 2e^{2z}\, dz$$

note that $f(z) = z^2 - 2e^{2z}$ is continuous on $\mathbb{C}$

by path independence I can choose γ



Method (A) parametrize $\quad \gamma: z(t) = it \qquad 1 \le t \le 2$
$\quad z'(t) = i$

$$z^2 - 2e^{2z} = (it)^2 - 2e^{2it} = -t^2 - 2e^{2ti}$$

$$\int_i^{2i} (z^2 - 2e^{2z})\, dz = \int_1^2 (-t^2 - 2e^{2ti})\cdot i\, dt = -i\left(\frac{t^3}{3} + ie^{2ti}\right)\Big|_1^2$$
$$= -i\left(\frac{8}{3} + ie^{4i} - \frac{1}{3} - ie^{2i}\right) = -\frac{7}{3}i - e^{4i} + e^{2i}$$

Method (B) By Path independence Lemma, $f(z)$ has an antiderivative $\quad \frac{d}{dz}\left(\frac{z^3}{3} - e^{2z}\right) = z^2 - 2e^{2z}$

So we can use the F.T.C for contours

$$\int_i^{2i} (z^2 - 2e^{2z})\, dz = \left(\frac{z^3}{3} - e^{2z}\right)\Big|_i^{2i} = \left(\frac{(2i)^3}{3} - e^{4i}\right) - \left(\frac{i^3}{3} - e^{2i}\right)$$

$$= \frac{-i8}{3} - e^{4i} + \frac{i}{3} + e^{2i}$$

$$= -\frac{7}{3}i - e^{4i} + e^{2i}$$