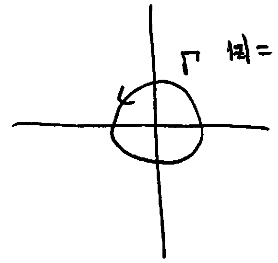
Assignment #5 - Solutions

1 (a) $\displaystyle\int_\Gamma \frac{e^z}{z^2(z-i)}\,dz$

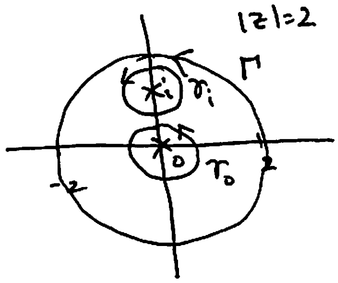


note singularities occur at

$z=0$ and $z=i$

Since $z=i$ is on the contour $|i|=1$

can't integrate



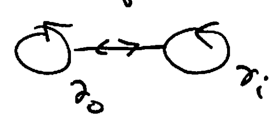
We use deformation invariance

D is the domain that includes Γ, γ_i, and γ_0

but not the points $z=0$ & $z=i$

since $f(z)=\dfrac{e^z}{z^2(z-i)}$ is analytic in D

and Γ can be deformed continuously into



$$\int_\Gamma \frac{e^z}{z^2(z-i)}\,dz = \underset{①}{\int_{\gamma_0} \frac{e^z/z-i}{z^2}\,dz} + \underset{②}{\int_{\gamma_i} \frac{e^z/z^2}{z-i}\,dz}$$

① $f(z)=\dfrac{e^z}{z-i}$ is analytic inside and on the contour γ_0

and $z=0$ is inside γ_0

We use Cauchy's Generalized Integral formula with $n=1$.

$$\int_{\gamma_0} \frac{e^z/z-i}{z^2}\,dz = \frac{2\pi i}{1!}\,\frac{d}{dz}\left(\frac{e^z}{z-i}\right)\Bigg|_{z=0} = 2\pi i\,\frac{(z-i)e^z-e^z}{(z-i)^2}\Bigg|_{z=0}$$

$$= 2\pi i\left(\frac{-i-1}{(-i)^2}\right) = 2\pi i\,(1+i)$$

(2) $f(z) = e^z/z^2$ is analytic inside and on the contour γ_i

and $z = i$ is inside γ_i

We use Cauchy's Integral formula.

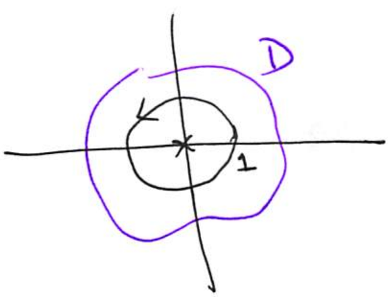
$$\int_{\gamma_i} \frac{e^z/z^2}{z-i}\,dz = 2\pi i \left.\frac{e^z}{z^2}\right|_{z=i} = 2\pi i \left(\frac{e^i}{i^2}\right) = -2\pi i e^i$$

so $$\int_{|z|=2} \frac{e^z}{z^2(z-i)}\,dz = 2\pi i\left(1+i-e^i\right) = 2\pi i - 2\pi - 2\pi i e^i$$

$$= 2\pi\left(i - 1 - ie^i\right)$$

#1 (b) $\displaystyle\int_\gamma \frac{\sin z}{z(z^2+2)}\,dz$ singularities occur at

$z=0$ and $z^2=-2$ or $z = \pm i\sqrt{2}$



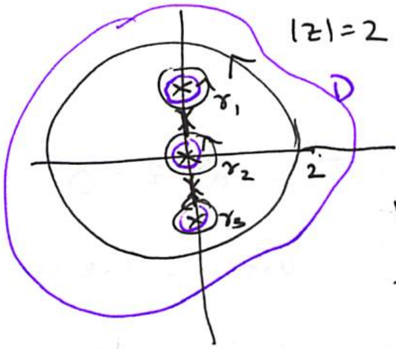
We use the Cauchy integral formula

i.e. $f(z) = \sin z / (z^2+2)$ is analytic in D

D is singly connected and z_0 is inside $|z|=1$

$$\int_{|z|=1} \frac{\sin z}{z(z^2+2)}\,dz = \int_{|z|=1} \frac{\sin z/(z^2+2)}{z}\,dz = 2\pi i \left.\frac{\sin z}{z^2+2}\right|_{z=0} = 0$$

1 (b)



there are 3 singularities in the interior of the curve $|z|=2$

$z_1 = i\sqrt{2}$, $z_2 = 0$, and $z_3 = -i\sqrt{2}$

We use deformation invariance to replace the curve $|z|=2$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$

that is $f(z) = \dfrac{\sin z}{z(z^2+2)}$ is analytic in The domain D

that contains the curves $|z|=2$, γ_1, γ_2, and γ_3.

but not z_1, z_2, and z_3.

We write

$$\int_{|z|=2} \frac{\sin z}{z(z+i\sqrt{2})(z-i\sqrt{2})}\,dz = \int_{\gamma_1} \frac{\frac{\sin z}{z(z+i\sqrt{2})}}{z-i\sqrt{2}}\,dz + \int_{\gamma_2} \frac{\frac{\sin z}{z^2+2}}{z}\,dz$$

$$+ \int_{\gamma_3} \frac{\frac{\sin z}{z(z-i\sqrt{2})}}{z+i\sqrt{2}}\,dz$$

and use Cauchy's Integral formula for each of the three integrals as in part (a)

$$= 2\pi i \left.\frac{\sin z}{z(z+i\sqrt{2})}\right|_{z=i\sqrt{2}} + 2\pi i \left.\frac{\sin z}{z^2+2}\right|_{z=0} + 2\pi i \left.\frac{\sin z}{z(z-i\sqrt{2})}\right|_{z=-i\sqrt{2}}$$

$$= \frac{2\pi i \sin(i\sqrt{2})}{i\sqrt{2}\,(2i\sqrt{2})} + 0 + \frac{2\pi i \sin(-i\sqrt{2})}{-i\sqrt{2}\,(-2i\sqrt{2})}$$

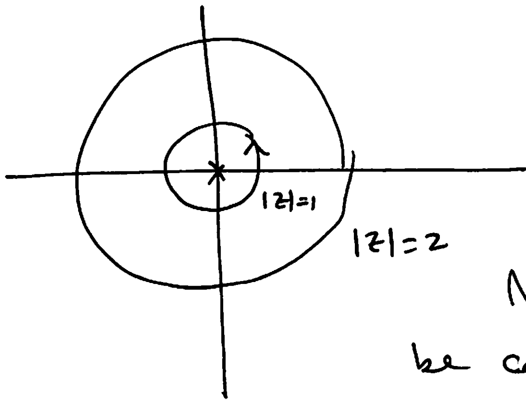
$= 0$ since $\sin z$ is an odd function

\# 1 (c) $\displaystyle\int_{\Gamma} \frac{\cosh z}{z^3}\, dz$

the only singularity is at $z = 0$

$\cosh z$ is analytic everywhere in $\mathbb{C}$

We use the generalized Cauchy integral with $n = 2$



Note that since both contour can be continuously transformed into one another in the domain D that includes both curves but not $z = 0$

$$\int_{|z|=1} \frac{\cosh z}{z^3}\, dz = \int_{|z|=2} \frac{\cosh z}{z^3}\, dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2}(\cosh z)\Big|_{z=0}$$

$$= \pi i \cosh z \Big|_{z=0} = \pi i$$

\#2 - We write

$$f(z) = \frac{1}{z} = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \dots$$

	$f^{(n)}(z)$	at $z=1$	
$n=0$	$\frac{1}{z}$	1	$0!$
$n=1$	$-\frac{1}{z^2}$	-1	$-1!$
$n=2$	$\frac{2}{z^3}$	$2\cdot 1$	$2!$
$n=3$	$-\frac{3\cdot 2}{z^4}$	$-3\cdot 2\cdot 1$	$-3!$
	$\vdots$		
	$(-1)^n \frac{n\cdot n-1\cdots\cdots n}{z^{n+1}}$	$(-1)^n n!$	

$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots$$

$$= \sum_{j=0}^{\infty} (-1)^j (z-1)^j$$

To find the radius of convergence, we compute

$$\lim_{j\to\infty} \left|\frac{(-1)^{j+1}(z-1)^{j+1}}{(-1)^j (z-1)^j}\right| = |z-1| < 1$$

$R=1$; the series converge to $f(z)$ in $0<|z-1|<1$

note that convergence is uniform in any closed sub-annulus.

#3. Consider $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$

This is a geometric series - it will converges provided $\left|\frac{z}{1+z}\right| < 1$ or $|z| < |z+1|$

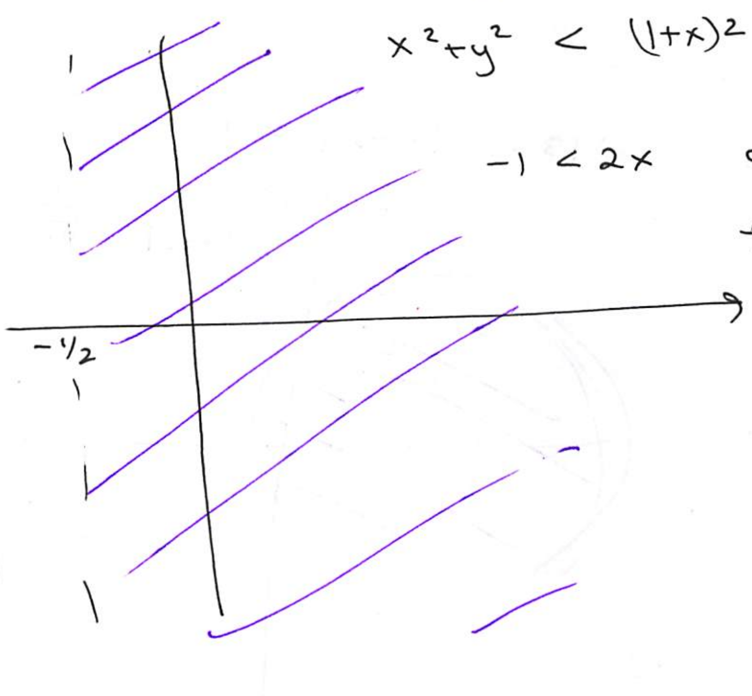
or $|z|^2 < |z+1|^2$

We write $z = x+iy$ and convert to cartesian coordinates

$$x^2+y^2 < (1+x)^2 + y^2 = 1+2x+x^2+y^2$$

$-1 < 2x$ or $x > -\frac{1}{2}$.

there is no conditions on y.



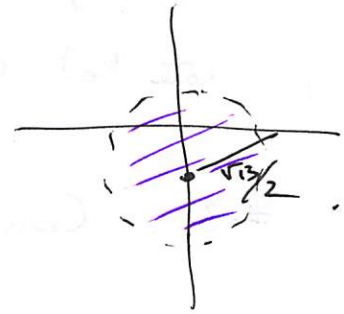
#4 (a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k (z+i)^k}{(2+3i)^k} = \sum_{k=0}^{\infty} \left(\frac{2(z+i)}{2+3i} \right)^k$ this is a geometric series.

we need $\left| \dfrac{2(z+i)}{2+3i} \right| < 1$ or $|2(z+i)| < |2+3i|$

$$|z+i| < \frac{|2+3i|}{2} = \frac{\sqrt{13}}{2}.$$

So the series converges in $|z+i| < \frac{\sqrt{13}}{2}$

this is a circle of radius $\frac{\sqrt{13}}{2}$ about $z = -i$



(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(3-i)^k}{k^2} (z+2)^k = \sum_{k=1}^{\infty} a^k (z+2)^k$ where $a_k = \dfrac{(3-i)^k}{k^2}$

We need $L = \displaystyle\lim_{k \to \infty} \left| \frac{a_{k+1} (z+2)^{k+1}}{a_k (z+2)^k} \right| < 1$

$$\lim_{k \to \infty} \left| \frac{(3-i)^{k+1} (z+2)^{k+1} \; k^2}{(k+1)^2 \; (z+2)^k \; (3-i)^k} \right| = \lim_{k \to \infty} \frac{|3-i| \, |z+2| \, \cancel{k^2}}{\cancel{k^2} \left(1 + \frac{1}{k}\right)^2} < 1$$

Since $\frac{1}{k} \to 0$.

we have $|3-i| \, |z+2| < 1$

$$|z+2| < \frac{1}{|3-i|} = \frac{1}{\sqrt{10}}$$

so $R = \sqrt{10}$

The series will converge in a circle of radius $\sqrt{10}$ about $z = -2$