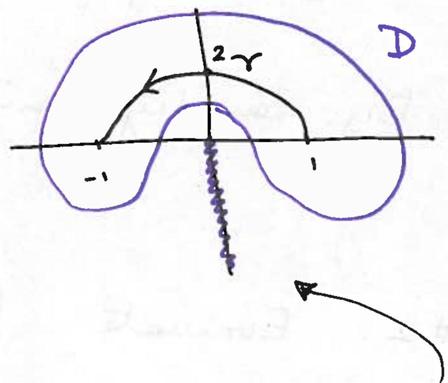
350, Winter 2016

Assignment #4 - Solutions

#1. Evaluate without performing an explicit computation

(a) $\displaystyle\int_\gamma \frac{1}{z}\,dz \qquad \gamma(t) = \cos t + 2i\sin t \qquad t \in [0,\pi]$



Note that $\frac{1}{z}$ is continuous and has an antiderivative in D

Take $\mathcal{L}_\tau(z) := \operatorname{Log} z + i \arg_\tau z$

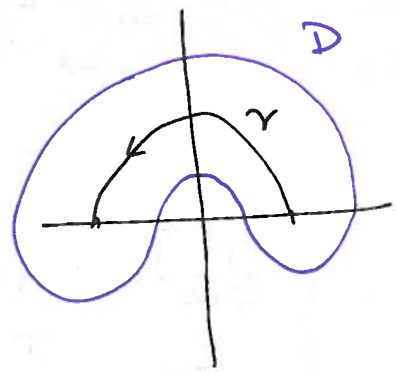
where $\tau < \arg_\tau(z) < \tau + 2\pi$ with $\tau = -\frac{\pi}{2}$

Then $\mathcal{L}_\tau$ is an antiderivative for $\frac{1}{z}$ in D

by the FTC for contours.

$$\int_\gamma \frac{1}{z}\,dz = \mathcal{L}_{-\frac{\pi}{2}}(z)\Big|_1^{-1} = \mathcal{L}_{-\frac{\pi}{2}}(-1) - \mathcal{L}_{-\frac{\pi}{2}}(1) = i\pi - 0 = i\pi$$

(b) $\displaystyle\int_\gamma \frac{1}{z^2}\,dz$,

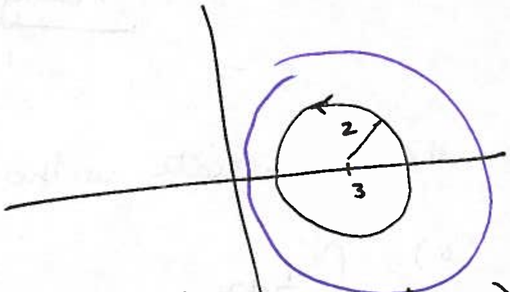


$\frac{1}{z^2}$ is continuous in D and has an antiderivative. $\frac{d}{dz}\left(\frac{-1}{z}\right) = \frac{1}{z^2}$

By the FTC for contours

$$\int_\gamma \frac{1}{z^2}\,dz = \frac{-1}{z}\Big|_1^{-1} = 1 + 1 = 2$$

(c) $\displaystyle\int_\gamma \frac{e^z}{z}\,dz$ $\quad \gamma = 3 + 2e^{it} \quad t \in [0, 2\pi]$

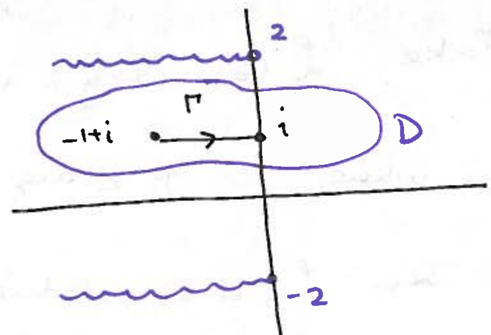


both e^z and z are analytic in D so $\frac{e^z}{z}$ is analytic in D

D is simply connected & γ is a loop (a closed contour) in D

By Cauchy's integral theorem $\displaystyle\int_\gamma \frac{e^z}{z}\,dz = 0$

#2. Evaluate $\displaystyle\int_\Gamma \frac{dz}{z^2+4}$



$z^2+4 = (z+2i)(z-2i)$

we use the method of partial fraction to write

$$\frac{1}{z^2+4} = \frac{A}{z+2i} + \frac{B}{z-2i} = \frac{A(z-2i) + B(z+2i)}{z^2+4} = \frac{(A+B)z + 2i(B-A)}{z^2+4}$$

so $A+B=0 \Rightarrow A=-B$

and $2i(B-A) = 1$
$2i(2B) = 1$
$-4B = i$ $\quad \Rightarrow B = \frac{-i}{4}$ & $A = \frac{i}{4}$

$$\int_\Gamma \frac{1}{z^2+4}\,dz = \int_\Gamma \frac{i}{4(z+2i)}\,dz - \int_\Gamma \frac{i}{4(z-2i)}\,dz$$

① $\mathrm{Log}(z+2i)$ is analytic in $\mathbb{C} \setminus \{x+iy \mid x \le 0 \text{ and } y=-2\}$

② $\mathrm{Log}(z-2i)$ is analytic in $\mathbb{C} \setminus \{x+iy \mid x \le 0 \text{ and } y = 2\}$

$x+iy-2i$
$x+i(y-2)$ so $x \le 0$ and $y-2=0$.

Choose D so that ① & ② are antiderivatives of $\frac{1}{z+2i}$ & $\frac{1}{z-2i}$

By the F.T.C. for contours.

$$\int_\Gamma \frac{dz}{z^2+4} = \frac{i}{4}\operatorname{Log}(z+2i)\Big|_{-1+i}^{i} - \frac{i}{4}\operatorname{Log}(z-2i)\Big|_{-1+i}^{i}$$

$$= \frac{i}{4}\left[\operatorname{Log}(3i) - \operatorname{Log}(-1+3i) - \operatorname{Log}(-i) + \operatorname{Log}(-1-i)\right]$$

$$\operatorname{Log}(3i) = \operatorname{Log}|3| + \frac{i\pi}{2}$$

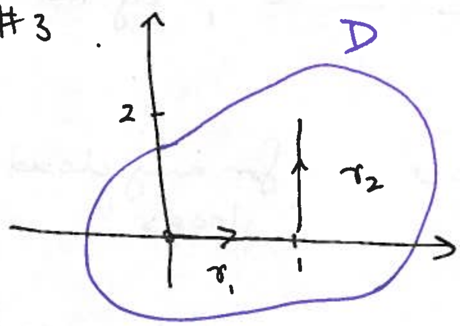
$$\operatorname{Log}(-1+3i) = \operatorname{Log}(\sqrt{10}) + i\operatorname{Arg}(-1+3i)$$

$$\operatorname{Log}(-i) = \cancel{\operatorname{Log} 1}^{\to 0} + i\left(-\frac{\pi}{2}\right)$$

$$\operatorname{Log}(-1-i) = \operatorname{Log}(\sqrt{2}) + i\left(-\frac{3\pi}{4}\right)$$

$$= \frac{i}{4}\left[\operatorname{Log} 3 - \operatorname{Log}\sqrt{10} - \operatorname{Log}\sqrt{2} + i\left(\pi - \frac{3\pi}{4} - \operatorname{Arg}(-1+3i)\right)\right]$$

#3.



$\Gamma = \{\gamma_1, \gamma_2\}$

$\gamma_1 : z_1(t) = t \qquad 0 \le t \le 1$

$z_1'(t) = 1$

$\gamma_2 : z_2(t) = 1 + 2ti \qquad 0 \le t \le 1$

$z_2'(t) = 2i$

(a) Evaluate.

$$\int_\Gamma (z + 2\bar{z})\, dz = \int_{\gamma_1} (z+2\bar{z})\, dz + \int_{\gamma_2} (z+2\bar{z})\, dz$$

$$= \int_0^1 3t\, dt + \int_0^1 (1 + 2ti + 2 - 4ti)\, 2i\, dt$$

$$= \frac{3t^2}{2}\Big|_0^1 + 2i\int_0^1 (3 - 2ti)\, dt = \frac{3}{2} + 2i\left(3t - \frac{2t^2}{2}i\right)\Big|_0^1$$

$$= \frac{3}{2} + 6i + 2 = \frac{7}{2} + 6i$$

(b) No – the integral is not path independent

Assume for a contradiction that it is. Since

$f(z) = z + 2\bar{z}$ is continuous in D (In fact it is continuous for all $\mathbb{C}$)

by the path independence lemma

$$\oint_\Gamma f(z)\,dz = 0 \quad \forall \text{ loops in } D.$$

But Morera's theorem implies that $f(z) = z + 2\bar{z}$ is

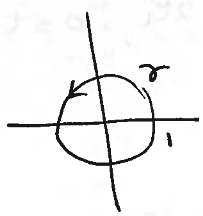
then Analytic in D – But this is false since $\bar{z}$

is nowhere analytic $\lightning$

Alternatively, Since f is continuous on $\mathbb{C}$, by the path independence lemma.

$$\oint_\gamma (z + 2\bar{z})\,dz = 0 \quad \text{for any closed loops}$$

pick



a circle of radius 1 around the origin

γ: $z(t) = e^{it}$ $\quad 0 \le t \le 2\pi$

$z'(t) = ie^{it}$

$$\oint_\gamma z + 2\bar{z}\,dz = \int_0^{2\pi} (e^{it} + 2e^{-it})\, ie^{it}\,dt$$

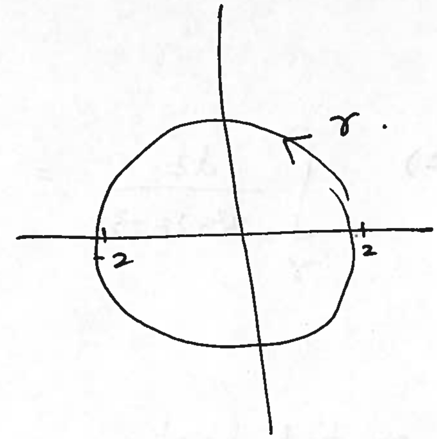
$$= i\int_0^{2\pi} (e^{2it} + 2)\,dt$$

$$= \cancel{\frac{e^{2it}}{2}\Big|_0^{2\pi}}^{\,0} + i2t\Big|_0^{2\pi} = 4\pi i$$

so $\int_\gamma z + 2\bar{z}\,dz$ is not path independent.

4. Let γ be as follows

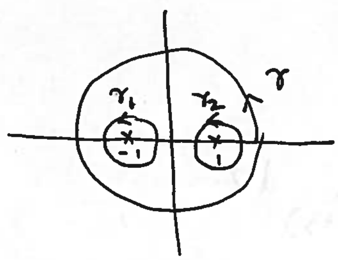
Compute.



(a) $\displaystyle\int_\gamma \frac{dz}{z^2-1} = \int \frac{dz}{(z+1)(z-1)}$

use partial fraction $\displaystyle\frac{1}{z^2-1} = \frac{A}{z-1} + \frac{B}{z+1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

or use Cauchy's integral formula.



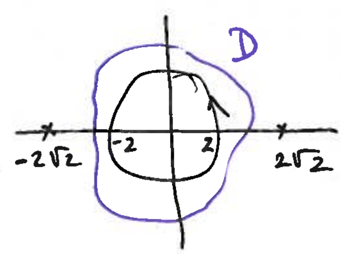
$$= \int_{\gamma_1} \frac{1/z-1}{z-(-1)}\, dz + \int_{\gamma_2} \frac{1/z+1}{z-1}\, dz$$

$$= 2\pi i \left(\frac{1}{z-1}\right)\Big|_{z=-1} + 2\pi i \left(\frac{1}{z+1}\right)\Big|_{z=1}$$

$$= 2\pi i \left(-\frac{1}{2}\right) + 2\pi i \left(\frac{1}{2}\right) = 0$$

(b) $\displaystyle\int_\gamma \frac{dz}{z^2-8}$ note that $z^2 = 8 \Rightarrow z = \pm\sqrt{8} = \pm 2\sqrt{2}$

by Cauchy's integral theorem,

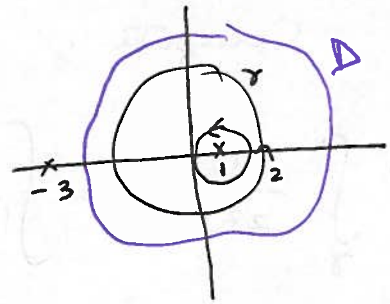


since $\displaystyle\frac{1}{z^2-8}$ is analytic in D

and D is simply connected.

$$\int_\gamma \frac{dz}{z^2-8} = 0$$

(c) $\displaystyle\int_\gamma \frac{dz}{z^2+2z-3} = \int_\gamma \frac{dz}{(z-1)(z+3)}$



use partial fractie

$$\frac{1}{(z-1)(z+3)} = \frac{A}{z-1} + \frac{B}{z+3} = \frac{A(z+3)+B(z-1)}{(z-1)(z+3)} = \frac{(A+B)z+(3A-B)}{(z-1)(z+3)}$$

$A+B=0 \Rightarrow A=-B$

$3A+A=1 \Rightarrow A=\frac{1}{4} \quad B=-\frac{1}{4}$

$$\int_\gamma \frac{dz}{z^2+2z-3}\,dz = \int_\gamma \frac{1}{4(z-1)}\,dz - \int_\gamma \frac{1}{4(z+3)}\,dz$$

$$= \frac{2\pi i}{4}$$

$\displaystyle\int_\gamma \frac{1}{z-1}\,dz = 2\pi i$

$=0$ since $\frac{1}{z+3}$ is analytic in D.