Math 350 W2016

Assignment # 1 - Solutions

Evaluate

#1. (a) $\left|\dfrac{i^9(2+i)^2}{(3+4i)^3}\right| = \dfrac{|i|^9\,|2+i|^2}{|3+4i|^3}$

$|i| = 1$, $|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$, $|3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

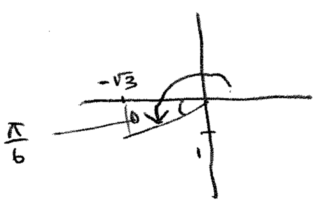
$$= \frac{(\sqrt{5})^2}{5^3} = \frac{5}{5^3} = \frac{1}{25}$$

(b) $\mathrm{Arg}\left((-\sqrt{3}-i)^2\right)$

Consider $z = -\sqrt{3} - i$

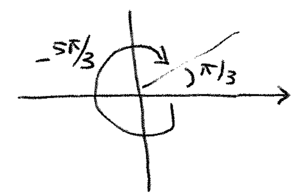
$|z| = \sqrt{3+1} = 2$

$\mathrm{Arg}\, z = -\dfrac{5\pi}{6}$



$z = 2e^{-i5\pi/6}$

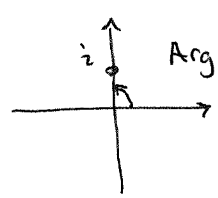
$\Rightarrow$ $z^2 = 4e^{-i5\pi/3}$



so $\mathrm{Arg}(z^2) = \dfrac{\pi}{3}$

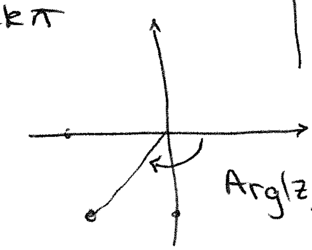
(c) $\arg\left(\dfrac{i}{-2-2i}\right)$ we use the fact that $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

set $z_1 = i$ $\Rightarrow$ $\arg i = \frac{\pi}{2} + 2k\pi$



$\mathrm{Arg}(z_1) = \frac{\pi}{2}$

set $z_2 = -2-2i$ $\Rightarrow$ $\arg(-2-2i) = -\dfrac{3\pi}{4} + 2k\pi$



$\mathrm{Arg}(z_2) = -\dfrac{3\pi}{4}$

so $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\pi}{2} - \left(\dfrac{-3\pi}{4}\right) + 2k\pi$

$= \dfrac{5\pi}{4} + 2k\pi$

or

$= -\dfrac{3\pi}{4} + 2k\pi$

where $\mathrm{Arg}\left(\dfrac{z_1}{z_2}\right) = -\dfrac{3\pi}{4}$

#2. Prove that $(\bar{z})^k = \overline{(z^k)}$ for every integer k provided $z \neq 0$ when k is negative.

* we can assume the following properties
1. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
2. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad z \neq 0$

First we prove the case $k \geq 0$ by induction

For $k=0$ $\quad (\bar{z})^0 = 1$ and $\overline{(z^0)} = \bar{1} = 1$ so this is true

for $k=1$ $\quad (\bar{z})^1 = \bar{z} = \overline{(z^1)}$ trivially true

for $k=2$ $\quad (\bar{z})^2 = \bar{z}\bar{z} = \overline{zz} = \overline{(z^2)}$

Assume it is true for k, consider

$$(\bar{z})^{k+1} = \bar{z}^k \cdot \bar{z} = \overline{(z^k)} \cdot \bar{z} \quad \text{by assumption}$$

$$= \overline{(z^k \cdot z)} \quad \text{by property 1.}$$

$$= \overline{(z^{k+1})} \quad ✓$$

We use this result together with property 2. to get the result for $k < 0$

That is if $k < 0$, then $(\bar{z})^k = (\bar{z})^{-j}$ where $j = -k > 0$

so $(\bar{z})^k = (\bar{z})^{-j} = \frac{1}{(\bar{z})^j}$ but from our result above $(\bar{z})^j = \overline{(z^j)}$

so $$= \frac{1}{\overline{(z^j)}} = \overline{\left(\frac{1}{z^j}\right)} \quad \text{by property 2}$$

$$= \overline{(z^{-j})}$$

$$= \overline{(z^k)} \quad ✓$$

#3. Prove that for all z,

(a) $e^{z+\pi i} = -e^{z}$

$$e^{z+\pi i} = e^{z}e^{\pi i} = e^{z}(\overset{-1}{\cancel{\cos\pi}} + i\overset{0}{\cancel{\sin\pi}}) = -e^{z}$$

(b) $\overline{e^{z}} = e^{\bar{z}}$

We use the fact that $e^{z} = e^{x}(\cos y + i\sin y)$ for $z = x+iy$

$$\overline{e^{z}} = e^{x}(\cos y - i\sin y)$$

$$= e^{x}(\cos y + i\sin(-y))$$ since $\sin\theta$ is an odd function

$$= e^{x}(\cos(-y) + i\sin(-y))$$ since $\cos\theta$ is an even function

$$= e^{x}e^{-iy} = e^{x-iy}$$

$$= e^{\bar{z}}$$

#4. Prove that if $|z| = 1$ $(z \neq 1)$, then

$$\operatorname{Re}\left[\frac{1}{1-z}\right] = \frac{1}{2}.$$

We use the fact that $\operatorname{Re} w = \dfrac{w+\bar{w}}{2}$ with $w = \dfrac{1}{1-z}$

$$\operatorname{Re}\left[\frac{1}{1-z}\right] = \frac{\frac{1}{1-z} + \overline{\left(\frac{1}{1-z}\right)}}{2} = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1-\bar{z}}\right)$$

$$= \frac{1}{2}\,\frac{1-\bar{z}+1-z}{(1-z)(1-\bar{z})}$$

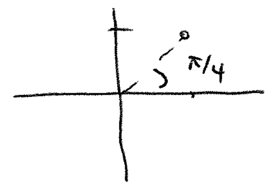
$$= \frac{1}{2}\,\frac{2-z-\bar{z}}{1-\bar{z}-z+z\bar{z}}$$ but $z\bar{z} = |z|^2 = 1$

$$= \frac{1}{2}\,\frac{2-z-\bar{z}}{2-\bar{z}-z}$$

$$= \frac{1}{2}$$

#5. Write in the polar form $re^{i\theta}$

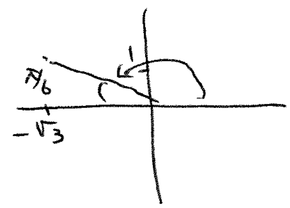
(a) $\dfrac{2+2i}{-\sqrt{3}+i}$



let $z_1 = 2+2i$; $|z_1| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$\mathrm{Arg}(z_1) = \frac{\pi}{4}$

so $z_1 = 2\sqrt{2}\, e^{i\pi/4}$



let $z_2 = -\sqrt{3}+i$; $|z_2| = \sqrt{3+1} = 2$

$\mathrm{Arg}(z_2) = \frac{5\pi}{6}$

So $z_2 = 2e^{i5\pi/6}$

$$\frac{z_1}{z_2} = \frac{2\sqrt{2}\,e^{i\pi/4}}{2\,e^{i5\pi/6}} = \sqrt{2}\, e^{i\left(\frac{\pi}{4}-\frac{5\pi}{6}\right)} = \sqrt{2}\, e^{-i\frac{7\pi}{12}}$$

(b) $$\frac{2i}{3e^{4+i}} = \frac{2e^{i\pi/2}}{3e^4 e^i} = \frac{2}{3e^4}\, e^{i\left(\frac{\pi}{2}-1\right)}$$

#6. Solve the equation $(z+1)^3 = z^3$.

We use the fact that $(z+1)^3 = z^3 \iff \left(\frac{z+1}{z}\right)^3 = 1$ for $z \neq 0$

set $w = \frac{z+1}{z}$ Then we need to find third roots of unity

$w^3 = 1$ or $w = 1^{1/3}$

solutions are $w_k = e^{\frac{2k\pi i}{3}}$ for $k = 0,1,2$

note that when $k=0 \Rightarrow w_0 = 1$ is not a solution

So the solution are $w_k = e^{\frac{2ki\pi}{3}}$ for $k = 1,2$

$w = \frac{z+1}{z}$

$w = 1+\frac{1}{z}$

$w-1 = \frac{1}{z}$

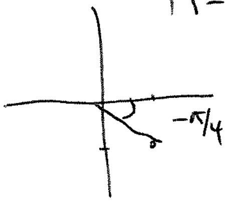
$z = \frac{1}{w-1}$

$$z_1 = \frac{1}{e^{\frac{i2\pi}{3}}-1}, \quad z_2 = \frac{1}{e^{\frac{i4\pi}{3}}-1}$$

7. Find all the values for the following expressions.

(a) $\sqrt{1-i}$ - first we transform into exponential notation

$|1-i| = \sqrt{1+1} = \sqrt{2}$



$\text{Arg } z = -\frac{\pi}{4}$

so $1-i = \sqrt{2}\, e^{-i\left(\frac{\pi}{4} + 2k\pi\right)}$

$$(1-i)^{1/2} = 2^{1/4}\left(e^{-i\frac{\pi+8k\pi}{4}}\right)^{1/2} \qquad k = 0, 1$$

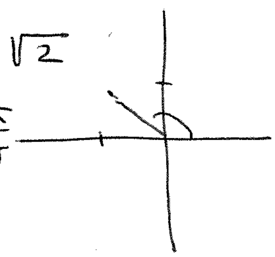
$$= 2^{1/4}\left(e^{-i\frac{\pi+8k\pi}{8}}\right)$$

$$z_0 = 2^{1/4} e^{-i\pi/8}, \quad z_1 = 2^{1/4} e^{-i9\pi/8}$$

(b) $(-1+i)^7$ in polar form

$|-1+i| = \sqrt{2}$

$\text{Arg } z = \frac{3\pi}{4}$

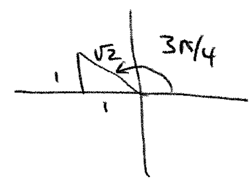


$$(-1+i)^7 = \left(2^{1/2} e^{i\frac{3\pi}{4}}\right)^7 = 2^{7/2}\left(\cos\frac{21\pi}{4} + i\sin\frac{21\pi}{4}\right)$$

but $\frac{21\pi}{4} = \frac{24-3}{4}\pi = 6\pi - \frac{3\pi}{4}$

we use periodicity to simplify our solution

$$= 2^{7/2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$



$$= 2^{7/2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= 2^{7/2}\left(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= 2^{6/2}\, 2^{1/2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

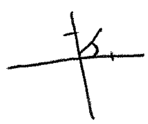
$$= -8(1+i)$$

(c) $\left(\dfrac{2i}{1+i}\right)^{1/3} =$

$2i = 2e^{i\pi/2}$

We start by transforming into polar form using

$|1+i| = \sqrt{2}$



$\text{Arg}(1+i) = \pi/4$

$$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

so $$\left(\frac{2i}{1+i}\right)^{1/3} = \left(\sqrt{2}\, e^{i\pi/4 + 2k\pi}\right)^{1/3}$$

$$= 2^{1/6}\, e^{i\frac{\pi+8k\pi}{12}} \quad \text{for } k = 0, 1, 2.$$