Math 350 - W13

Test # 1

#1. Let $f(z) = \dfrac{z^2}{|z^2|}$. Does f have a limit at $z \to 0$?

Set $z = x + iy$

We have $z^2 = x^2 - y^2 + i2xy$

$$|z^2| = |z|^2 = x^2 + y^2$$

So $f(z) = \dfrac{x^2 - y^2 + i2xy}{x^2+y^2}$.

Ⓐ Take limit as $z \to 0$ along the y-axis $(x=0)$

$$\lim_{z \to 0} f(z) = \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$$

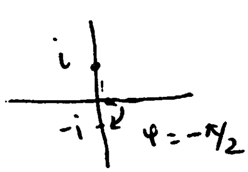
Ⓑ Take limit along the x-axis $(y=0)$

$$\lim_{z\to 0} f(z) = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

Since the two limits are different, the limit as $z \to 0$ doesn't exist.

#2. (a) Solve for z: $z^4 + i = 0$. Express your answer in exponential form

$$z^4 + i = 0 \iff z^4 = -i \iff z = (-i)^{1/4}$$



so $-i = e^{-i\pi/2}$ The roots are $z_k = e^{i(-\frac{\pi}{2} + 2k\pi)1/4} = e^{i\pi(-\frac{\pi}{8} + \frac{4k\pi}{8})}$

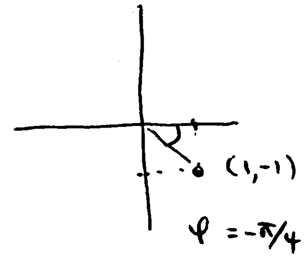
for $k = 0, 1, 2, 3$

$$z_0 = e^{-i\pi/8}$$
$$z_1 = e^{i3\pi/8}$$
$$z_2 = e^{i7\pi/8}$$
$$z_3 = e^{i11\pi/8}$$

2(b). Evaluate $\left(\frac{1}{1+i}\right)^{10}$

We start by transforming z into exponential form.

We have $z = \frac{1}{1+i} = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2} = \frac{1}{2}(1-i)$



$$|z| = \frac{1}{2}\sqrt{1+1} = \frac{\sqrt{2}}{2}$$

$$\text{so } z = \frac{\sqrt{2}}{2} e^{-i\pi/4} = \frac{1}{\sqrt{2}} e^{-i\pi/4} = 2^{-1/2} e^{-i\pi/4}$$

$$\text{so } \left(\frac{1}{1+i}\right)^{10} = \left(2^{-1/2} e^{-i\pi/4}\right)^{10}$$

$\Rightarrow \frac{5\pi}{2} = \frac{4\pi}{2} + \frac{\pi}{2} = 2\pi +$

$$= 2^{-5} e^{-i10\pi/4} = \frac{1}{32} e^{-i5\pi/2}$$

$$= \frac{1}{32} e^{-i2\pi} e^{-i\pi/2}.$$

$$= \frac{1}{32} e^{-i\pi/2}$$

In polar form

$$= \frac{1}{32}\left(\cos(-\pi/2) + i \sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{32}\left(\cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\right)$$

$$= -\frac{i}{32}$$

(c) Find all distinct values of $1^{3/4}$. Simplify your answer

$$1^{3/4} = \left(e^{i2k\pi}\right)^{3/4} = e^{\frac{i3\pi k}{2}} \qquad k = 0, 1, 2, 3$$

$k=0$, $z_0 = 1$

$k=1$, $z_1 = e^{i3\pi/2} = \cancel{\cos\frac{3\pi}{2}}^{\,0} + i \sin\frac{3\pi}{2} = -i$

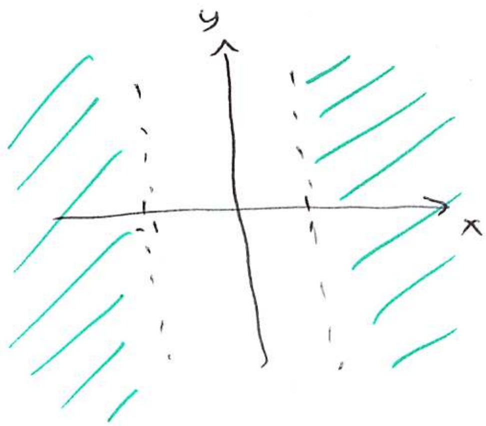
$k=2$, $z_2 = e^{i3\pi} = \cos 3\pi + i \cancel{\sin 3\pi}^{\,0} = -1$

$k=3$, $z_3 = e^{i9\pi/2} = e^{4\pi i} e^{i\pi/2} = e^{i\pi/2}$

$= \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i$

\#3. Describe the set of points $z \in \mathbb{C}$ satisfying the following equations and determine which of these set is a domain.

(a) $(\text{Re}\, z)^2 > 1$



Let $z = x + iy$

$(\text{Re}\, z) = x \Rightarrow (\text{Re}\, z)^2 = x^2 > 1$

note that there are no restriction on y but $x > 1$ and $x < -1$

the resulting set is open, but not connected, therefore not a domain

(b) $\text{Re}(\bar{z} - i) = 2$

Let $z = x + iy$

$\text{Re}(\bar{z} - i) = \text{Re}(x - iy - i)$

$= x$

$\Rightarrow$ the line $x = 2$ is connected, but not open so not a domain

\#4. Use the formal definition of limits to show that the function $f(z) = \frac{iz}{2}$ is continuous at $z = i$.

We need to show that $\lim_{z \to i} f(z) = f(i) = \frac{i \cdot i}{2} = \frac{-1}{2}$

Fix $\varepsilon > 0$, we claim that there exist an $\delta > 0$ such that

$$|f(z) - f(i)| < \varepsilon \quad \text{whenever} \quad |z - i| < \delta$$

Consider $|f(z) - f(i)| = \left|\frac{iz}{2} + \frac{1}{2}\right| = \frac{1}{2}|i(z - i)| = \frac{1}{2}|z - i| < \frac{\delta}{2}$

Choose $\delta = \varepsilon$. Then it follows that

$$|f(z) - f(i)| = \tfrac{1}{2}|z-i| < \tfrac{1}{2}\delta = \tfrac{1}{2}\varepsilon < \varepsilon$$

$\boxtimes$

#5- Derive the Cauchy Riemann equations, and explain why they give a necessary conditions for the existence of $f'(z)$ at $z = z_0$.

Let $f(z) = u(x,y) + i\,v(x,y)$

Consider $\displaystyle\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0)$

The derivative exists, provided this limit exists, which means that the limit exists no matter which path is used to let $\Delta z \to 0$. The Cauchy-Riemann equations are derived by (A) letting $\Delta x \to 0$ first, then $\Delta y \to 0$ and (B) letting $\Delta y \to 0$ first, then $\Delta x \to 0$. A necessary condition for the limit above to exist is that each of these limit must be the same.

We have

$$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) + i\left(v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\right)}{\Delta x + i\Delta y}$$

(A) let $\Delta x \to 0$; then $\Delta y \to 0$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0) + i\left(v(x_0,y_0+\Delta y) - v(x_0,y_0)\right)}{i\Delta y}$$

$$= -i \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}$$

(B) let $\Delta y \to 0$; Then $\Delta x \to 0$

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0) + i\left(v(x_0+\Delta x, y_0) - v(x_0,y_0)\right)}{\Delta x}$$

$$= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}$$

Equate the real & imaginary part in (A) + (B)

this yields

$$\left.\begin{aligned} &(1) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ &(2) \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{aligned}\right\}$$ these are the Cauchy-Riemann Equations

#6. Consider $f(z) = z^3 + iz^2 - z$.

(a) Write $f(z)$ in the form $u(x,y) + iv(x,y)$

Let $z = x + iy$

$$z^2 = x^2 - y^2 + i2xy$$

$$z^3 = (x+iy)(x^2-y^2+i2xy)$$

$$= x^3 - y^2x + i2x^2y + iyx^2 - iy^3 - 2xy^2$$

$$= x^3 - 3y^2x + i(3x^2y - y^3)$$

So $f(z) = x^3 - 3y^2x + i(3x^2y - y^3) + ix^2 - iy^2 - 2xy - x - iy$

$$= x^3 - 3y^2x - 2xy - x + i\left(3x^2y - y^3 + x^2 - y^2 - y\right)$$

so $u(x,y) = x^3 - 3y^2x - 2xy - x$

$v(x,y) = 3x^2y - y^3 + x^2 - y^2 - y$.

(b) Show that $f(z)$ is entire

We need to show that the C.-R. Equations are satisfied for all $z \in \mathbb{C}$ and that u, v & their first partial derivative are continuous for all $z \in \mathbb{C}$

C.R. Equations $u_x = v_y$ & $u_y = -v_x$

$u(x,y) = x^3 - 3y^2x - 2xy - x$

$u_x = 3x^2 - 3y^2 - 2y - 1$

$u_y = -6yx - 2x$

$v(x,y) = 3x^2y - y^3 + x^2 - y^2 - y$

$v_x = 6xy + 2x$

$v_y = 3x^2 - 3y^2 - 2y - 1$

$u_y = -v_x$ ✓

$u_x = v_y$ ✓

u, v and all the first partial derivative are polynomials $\Rightarrow$ continuous for all $z \in \mathbb{C}$

c) Are the functions u and v in part (a) harmonic? Justify your answer

Since $f = u(x,y) + i v(x,y)$ is analytic for all $z \in \mathbb{C}$ then u & v are harmonic in $\mathbb{C}$