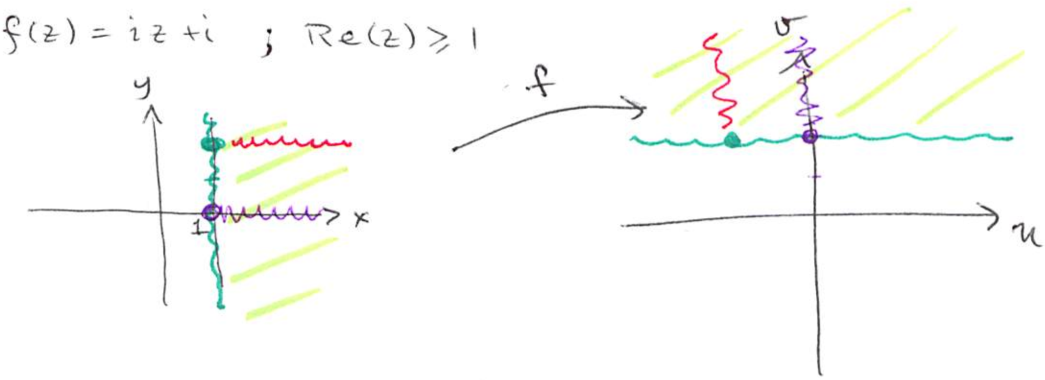
Assignment # 2 - Solutions

#1 Describe The range

(a) $f(z) = iz + i \quad ; \quad Re(z) \geq 1$



Let $z = x+iy$

$$\begin{aligned} f(z) &= i(z+1) \\ &= i(x+iy+1) \\ &= ix - y + i \\ &= -y + i(x+1) \end{aligned}$$

$y=2 \quad x \geq 1$

$= -2 + i(x+1)$

$y = 0 \quad x \geq 1$

$\to i(x+1)$

$x = 1 \Rightarrow i2$

$x = 1$

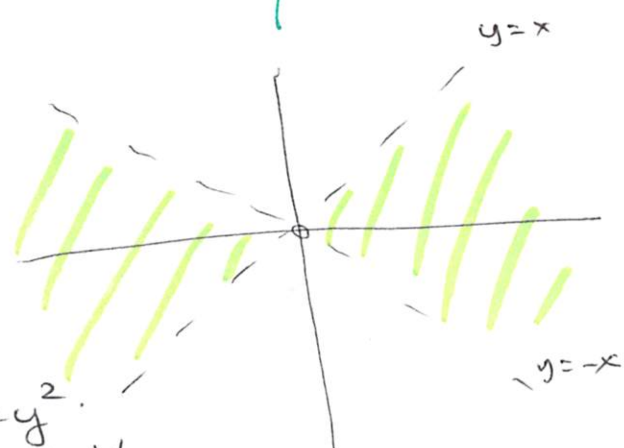
$y \in \mathbb{R}$

$\to -y + 2i$

so $y = 2 \to -2 + 2i$

see graph - note that the portion of the first quadrant gets shifted to the 2nd Quadrant; likewise, the portion of the pre-image in The 4th Quadrant gets shifted to The 1st Quadrant

(b) $\{ z \in \mathbb{C} \mid Re(z^2) > 0 \}$



Set $z = x + iy$

$z^2 = x^2 - y^2 + i2xy$

so $Re(z^2) = x^2 - y^2 > 0$

the boundary is $x^2 = y^2$.

the point $(2,1)$ is in but $(1,2)$ isn't.

the lines $y = x$ & $y = -x$ are not included, neither is the origin since it would mean $0 > 0$.

\# 2. Use the formal definition of limits to show that $\displaystyle\lim_{z\to i}\frac{z^2+1}{z-i} = 2i$.

Proof: Let $\varepsilon>0$, We need to show that there exist a $\delta>0$ such that $|f(z)-w_0|<\varepsilon$ whenever $0<|z-z_0|<\delta$.
where $z_0=i$ & $w_0=2i$

$$|f(z)-w_0| = \left|\frac{z^2+1}{z-i}-2i\right| = \left|\frac{(z-i)(z+i)}{z-i}-2i\right|$$

$$= |z+i-2i|$$

$$= |z-i| < \delta$$

Choose $\delta=\varepsilon$ then $|f(z)-w_0| = |z-i| < \delta = \varepsilon$

\# 3. By def$^{\underline{n}}$,

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\frac{\operatorname{Im}(z_0+\Delta z)-\operatorname{Im}(z_0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\frac{\operatorname{Im}(z_0)+\operatorname{Im}(\Delta z)-\operatorname{Im}(z_0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\frac{\operatorname{Im}(\Delta z)}{\Delta z}$$

set $\Delta z=\Delta x+i\Delta y$; we look at the limit along two different paths.
along the horizontal $\Delta y=0$, $\Delta z=\Delta x$ then $\operatorname{Im}\Delta z=0$ so the limit $=0$
along the vertical $\Delta x=0$, $\Delta z=i\Delta y$, then $\operatorname{Im}\Delta z=\Delta y$ so the limit is $=\frac{1}{i}$. Since the limit from Two different path is different the derivative doesn't exist. But z_0 was arbitrary, so this function is nowhere differentiable $\Rightarrow$ nowhere analytic.

Let $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$.

#4 Use the def$^{\underline{n}}$ of limits to prove that

$$\lim_{z \to z_0} f(z) = w_0$$

iff $\displaystyle\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0$ and $\displaystyle\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0$.

$\Rightarrow$ Assume $\displaystyle\lim_{z \to z_0} f(z) = w_0$. We need to show that given $\varepsilon > 0$, there exist $\delta > 0$ such that $|u(x,y) - u_0| < \varepsilon$ and $|v(x,y) - v_0| < \varepsilon$ whenever $|(x,y) - (x_0,y_0)| < \delta$ or $|z - z_0| < \delta$.

Fix $\varepsilon > 0$, by assumption, $\exists\ \delta > 0 \ni |f(z) - w_0| < \varepsilon$ whenever $|z - z_0| < \delta$.

Consider

$$|u(x,y) - u_0| \le |u(x,y) - u_0 + i(v(x,y) - v_0)|$$
$$= |f(z) - w_0| < \varepsilon \quad \text{provided } |z - z_0| < \delta.$$

likewise

$$|v(x,y) - v_0| \le |u(x,y) - u_0 + i(v(x,y) - v_0)|$$
$$= |f(z) - w_0| < \varepsilon \quad \text{provided } |z - z_0| < \delta.$$

so $\delta > 0$ from assumption gives the desired result $\blacksquare$

$\Leftarrow$ Assume $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0$ and $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0$.

We need to show that given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(z) - w_0| = |u(x,y) - u_0 + i(v(x,y) - v_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.

Fix $\varepsilon > 0$,

We have

$$|f(z) - w_0| = |u(x,y) - u_0 + i(v(x,y) - v_0)|$$
$$\leq |u(x,y) - u_0| + |v(x,y) - v_0|$$

pick δ_1 so that $|u(x,y) - u_0| < \varepsilon/2$ whenever $|z - z_0| < \delta_1$

and δ_2 so that $|v(x,y) - v_0| < \varepsilon/2$ whenever $|z - z_0| < \delta_2$

then if $\delta = \min(\delta_1, \delta_2)$

$$|f(z) - w_0| \leq |u(x,y) - u_0| + |v(x,y) - v_0| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

whenever $|z - z_0| < \delta$ $\blacksquare$

#5 - Use Cauchy-Riemann's equations to show that $f(z) = x^2 + iy^2$ is nowhere analytic

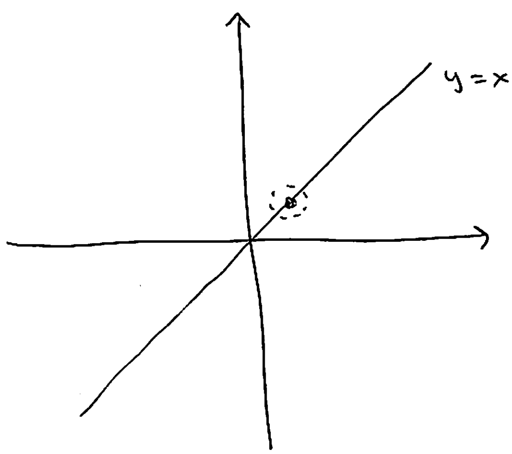
we have $\quad u(x,y) = x^2 \rightarrow u_x = 2x, \quad u_y = 0$

$\qquad\qquad v(x,y) = y^2 \rightarrow v_x = 0, \quad v_y = 2y$

We need

$u_x = v_y$ and $v_x = -u_y$ $\iff$ $2x = 2y$ or $x = y$.

so $f(z)$ is differentiable at points on the line $y = x$



but for any points on the line $y=x$, every neighborhood will contain points where f is not differentiable.

So f is nowhere analytic.