Assignment #5 - Solutions

#1 - We write

$$f(z) = \frac{1}{z} = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \cdots$$

$f^{(n)}(z)$	at $z=1$
$\frac{1}{z}$	$1 = 0!$
$-\frac{1}{z^2}$	$-1 = -1!$
$\frac{2}{z^3}$	$2\cdot 1 = +2!$
$-\frac{3\cdot 2}{z^4}$	$-3\cdot 2\cdot 1 = -3!$
$f^{(n)}(z)$	$= n!(-1)^n$

$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

$$= \sum_{j=0}^{\infty} (-1)^j (z-1)^j$$

To find the radius of convergence, we solve

$$\lim_{j\to\infty} \left| \frac{(-1)^{j+1}(z-1)^{j+1}}{(-1)^j (z-1)^j} \right| = |z-1| < 1 = R$$

This series will converge to $f(z)$ in $0 < |z-1| < 1$

Note that the convergence is uniform in any sub-annulus.

\# 2 Consider $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$

this is a geometric series; it will converge provided

$$\left|\frac{z}{1+z}\right| < 1 \quad \text{or} \quad |z| < |z+1|$$

$$|z|^2 < |z+1|^2$$

we write $z = x+iy$ and convert to cartesian coordinates

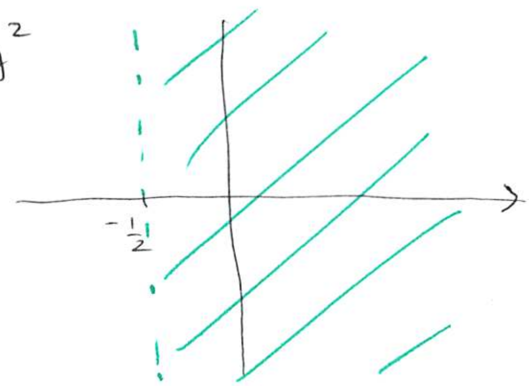
$$x^2+y^2 < (1+x)^2 + y^2$$

$$x^2+y^2 < 1+2x+x^2+y^2$$

$$-1 < 2x$$

$$\text{or} \quad x > \frac{-1}{2}$$

there are no conditions on y



\# 3 (a) $\displaystyle\sum_{k=0}^{\infty}\frac{2^k(z+i)^k}{(2+3i)^k} = \sum_{k=0}^{\infty}\left(\frac{2(z+i)}{2+3i}\right)^k$ this is a geometric series

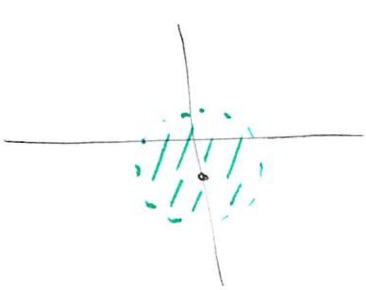
we need $\left|\dfrac{2(z+i)}{2+3i}\right| < 1$

or $|2(z+i)| < |2+3i|$

$$|z+i| < \frac{|2+3i|}{2} = \frac{\sqrt{2^2+3^2}}{2} = \frac{\sqrt{13}}{2}$$

This series will converge in $|z+i| < \dfrac{\sqrt{13}}{2}$

this is a circle of radius $\dfrac{\sqrt{13}}{2}$ about $z=-i$



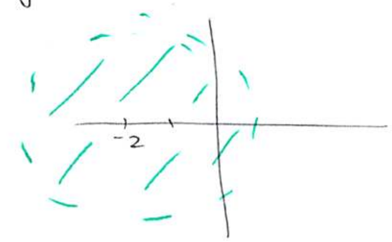
(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(3-i)^k}{k^2}(z+2)^k = \sum_{k=1}^{\infty} a^k (z+2)^k$ where $a^k = \dfrac{(3-i)^k}{k^2}$

we need $\displaystyle L = \lim_{k\to\infty} \left| \frac{a_{k+1}(z+2)^{k+1}}{a_k (z+2)^k} \right| < 1$

$$\lim_{k\to\infty} \left| \frac{(3-i)^{k+1}}{(k+1)^2} \frac{(z+2)^{k+1}}{(z+2)^k} \frac{k^2}{(3-i)^k} \right| = \lim_{k\to\infty} \frac{|3-i||z+2| k^2}{k^2(1+\frac{1}{k})^2} < 1$$

$$|z+2| < \frac{1}{|3-i|} = \frac{1}{\sqrt{10}} \Rightarrow R = \sqrt{10}$$

the series will converges in a circle of radius $\sqrt{10}$ about $z=-2$



#4. $\displaystyle f(z) = \frac{z}{z^2+4z-12} = \frac{z}{8}\left[\frac{1}{z-2} - \frac{1}{z+6}\right]$

(a) If $|z|<1$ then $\left|\frac{z}{2}\right|<1$ and $\left|\frac{z}{6}\right|<1$, we write

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-z/2} = -\frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j$$

$$\frac{1}{z+6} = \frac{1}{6}\frac{1}{1-(-z/6)} = \frac{1}{6}\sum_{j=0}^{\infty}(-1)^j\left(\frac{z}{6}\right)^j$$

so $\displaystyle f(z) = -\frac{z}{8}\left[\sum_{j=0}^{\infty}\frac{z^j}{2^{j+1}} + \sum_{j=0}^{\infty}\frac{(-1)^j z^j}{6^{j+1}}\right]$

$$= -\frac{1}{8}\sum_{j=0}^{\infty}\left[\frac{1}{2^{j+1}} + \frac{(-1)^j}{6^{j+1}}\right] z^{j+1}$$

set $j+1=k$

$j=0 \Rightarrow k=1$

$$= -\frac{1}{8}\sum_{k=1}^{\infty}\left[\frac{1}{2^k} + \frac{(-1)^{k+1}}{6^k}\right] z^k$$

(b) here $\frac{|z|}{6} < \frac{|z|}{4} < 1$ but $|z| > 3 \Rightarrow \frac{3}{|z|} < 1$

$\Rightarrow \frac{2}{|z|} < \frac{3}{|z|} < 1$

so
$$\frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{2}{z}\right)^j$$

$$\frac{1}{z+6} = \frac{1}{6}\,\frac{1}{1-\left(\frac{-z}{6}\right)} = \frac{1}{6}\sum_{j=0}^{\infty}(-1)^j\left(\frac{z}{6}\right)^j$$

so
$$f(z) = -\frac{z}{8}\left[\frac{1}{z}\sum_{j=0}^{\infty}\frac{2^j}{z^j} - \frac{1}{6}\sum_{j=0}^{\infty}(-1)^j\left(\frac{z}{6}\right)^j\right]$$

$$= -\frac{1}{8}\left[\sum_{j=0}^{\infty}\frac{2^j}{z^j} + \sum_{j=0}^{\infty}(-1)^{j+1}\frac{z^{j+1}}{6^{j+1}}\right]$$

$$= -\frac{1}{8}\left[\sum_{j=0}^{\infty}\frac{2^j}{z^j} + \sum_{j=1}^{\infty}(-1)^j\frac{z^j}{6^j}\right]$$

(c) Here, $\frac{2}{|z|} < \frac{6}{|z|} < \frac{8}{|z|} < 1$

so
$$\frac{1}{z-2} = \frac{1}{z}\left(\frac{1}{1-2/z}\right) = \frac{1}{z}\sum_{j=0}^{\infty}\frac{2^j}{z^j}$$

$$\frac{1}{z+6} = \frac{1}{z}\left(\frac{1}{1-\frac{-6}{z}}\right) = \frac{1}{z}\sum_{j=0}^{\infty}(-1)^j\frac{6^j}{z^j}$$

so
$$f(z) = -\frac{1}{8}\left[\sum_{j=0}^{\infty}\frac{2^j}{z^j} + \sum_{j=0}^{\infty}(-1)^j\frac{6^j}{z^j}\right]$$

$$= -\frac{1}{8}\sum_{j=0}^{\infty}\frac{2^j+(-1)^j 6^j}{z^j}$$