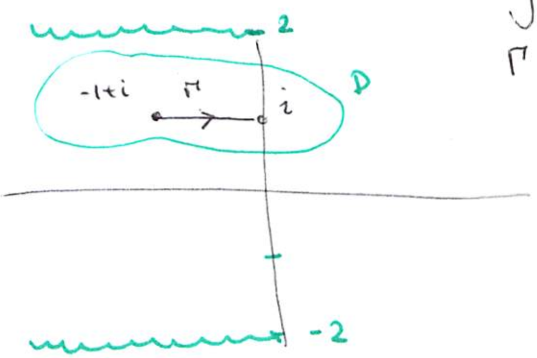
Math 350 W 2013

Assigment # 4 - Solutions

#1 Evaluate $\displaystyle\int_\Gamma \frac{dz}{z^2+4}$



note that $z^2+4 = (z+2i)(z-2i)$

we use the method of partial fracti
To isolate each factor

$$\frac{1}{z^2+4} = \frac{A}{z+2i} + \frac{B}{z-2i}$$

$$= \frac{A(z-2i) + B(z+2i)}{z^2+4}$$

$$1 = (A+B)z + 2i(B-A)$$

so $A+B=0 \Rightarrow A=-B$

and $2i(B-A) = 1$

$2i(2B) = 1$

$-4B = i$

$B = \frac{-i}{4}$ $\quad A = \frac{i}{4}$

$$\int_\Gamma \frac{dz}{z^2+4} = \int_\Gamma \frac{i}{4(z+2i)}\,dz - \int_\Gamma \frac{i}{4(z-2i)}\,dz$$

Log $(z+2i)$ is analytic in

$\mathbb{C} \setminus \{x+iy \mid x \le 0 \;\&\; y=-2\}$

Log $(z-2i)$ is analytic in

$\mathbb{C} \setminus \{x+iy \mid x \le 0 \;\&\; y=2\}$

$x+iy-2i$

$x+i(y-2)$

$x \le 0$, $y-2=0$.

$f_1(z) = \frac{1}{z+2i}$ is continuous in D and has antiderivative Log $(z+2i)$

$f_2(z) = \frac{1}{z-2i}$ is continuous in D and has antiderivative Log $(z-2i)$

By the F.T.C. $\displaystyle\int_\Gamma \frac{dz}{z^2+4} = \frac{i}{4}\,\mathrm{Log}(z+2i)\Big|_{-1+i}^{i} - \frac{i}{4}\,\mathrm{Log}(z-2i)\Big|_{-1+i}^{i}$

$$= \frac{i}{4}\left[\text{Log}(3i) - \text{Log}(-1+3i) - \text{Log}(-i) + \text{Log}(-1-i)\right]$$

$$\text{Log}(3i) = \text{Log}|3| + \frac{i\pi}{2}$$

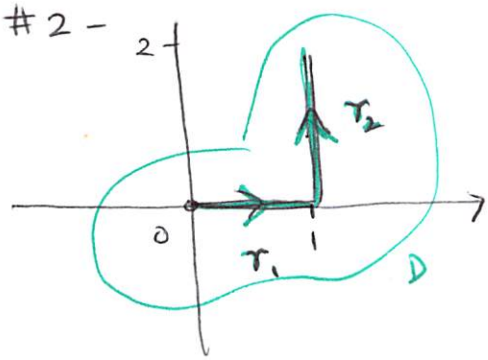
$$\text{Log}(-1+3i) = \text{Log}(\sqrt{10}) + i\,\text{Arg}(-1+3i)$$

$$\text{Log}(-i) = \cancel{\text{Log}}\overset{\nearrow 0}{1} + i\left(-\frac{\pi}{2}\right)$$

$$\text{Log}(-1-i) = \text{Log}\sqrt{2} + i\left(-\frac{3\pi}{4}\right)$$

$$= \frac{i}{4}\left[\text{Log}\,3 - \text{Log}\sqrt{10} - \text{Log}\sqrt{2} + i\left(\pi - \frac{3\pi}{4} - \text{Arg}(-1+3i)\right)\right]$$

#2 -



$\Gamma = \{\gamma_1, \gamma_2\}$

(a) Evaluate $\displaystyle\int_\Gamma z + 2\bar{z}\, dz$

$$= \int_{\gamma_1} z + 2\bar{z}\, dz + \int_{\gamma_2} z + 2\bar{z}\, dz$$

γ_1: $z_1(t) = t \qquad 0 \le t \le 1$

$z_1'(t) = 1$

$\gamma_2 = z_2(t) = 1 + i2t \qquad 0 \le t \le 1$

$z_2'(t) = i2$

$$= \int_0^1 (t + 2t)\, dt + \int_0^1 (1 + i2t + 2(1-2it))2i\,dt$$

$$= \frac{3t^2}{2}\Big|_0^1 + 2i\int_0^1 (3 - 2it)\, dt$$

$$= \frac{3}{2} + 2i(3t - it^2)\Big|_0^1 = \frac{3}{2} + 2i(3 - i)$$

$$= \frac{3}{2} + 6i + 2 = \frac{7}{2} + 6i$$

(b) No - the integral is not path independent.

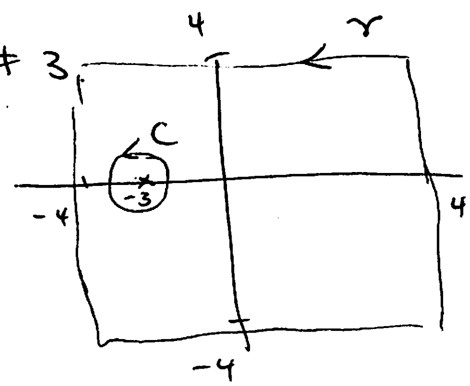
Assume for a contradiction that it is. since $f(z) = z + 2\bar{z}$ is continuous in D by the path independence lemma

$$\oint_{\Gamma} f(z)\,dz = 0 \quad \forall \text{ loops in } D.$$

But Morera's theorem implies that $f(z) = z + 2\bar{z}$ is then analytic in D - But this is false since $\bar{z}$ is nowhere analytic $\lightning$

So our Assumption was false - the integral is not path independent.

\# 3



Compute $\displaystyle\oint_{\gamma} \frac{z^n}{(z+3)^{n+1}}\,dz$

This function has a singularity at $z = -3$ so we use Cauchy's Generalized integral formula.

$\gamma \rightsquigarrow C$

$$\oint_{\gamma} \frac{z^n}{(z+3)^{n+1}}\,dz = \oint_{C} \frac{z^n}{(z+3)^{n+1}}\,dz = \frac{2\pi i}{n!} \frac{d^n}{dz^n}(z^n) \Big|_{z=-3}$$

z^n is analytic in and on C

$z = -3$ is in the interior of C

We compute the derivative

$$\frac{d}{dz} z^n = n z^{n-1}$$

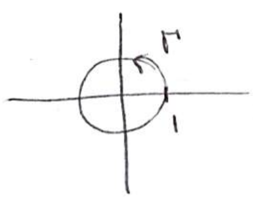
$$\frac{d^2}{dz^2} z^n = n(n-1) z^{n-2}$$

$$\vdots$$

$$\frac{d^n}{dz^n}(z^n) = n(n-1)\cdots 1\, z^{n-n} = n!$$

so $$\oint_\gamma \frac{z^n}{(z+3)^{n+1}}\, dz = \frac{2\pi i}{n!} \cdot n! = 2\pi i$$

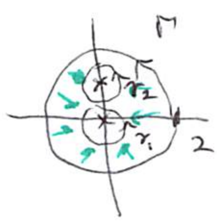
#4. (a)



$$\int_\Gamma \frac{e^z}{z^2(z-i)}\, dz$$

can't do because $z = i$ is a point on the contour.

$\Gamma : |z| = 1$



There are singularities at

$z = 0$ & $z = i$

we use deformation invariance so

$$\int_\Gamma F(z)\, dz = \int_{\gamma_1} F(z)\, dz + \int_{\gamma_2} F(z)\, dz$$

$\Gamma : |z| = 2$

$$\int_\Gamma \frac{e^z}{z^2(z-i)}\, dz = \int_{\gamma_1} \frac{e^z/z-i}{z^2}\, dz + \int_{\gamma_2} \frac{e^z/z^2}{z-i}\, dz$$

① ②

① $f(z) = \dfrac{e^z}{z-i}$ is analytic inside and on the contour γ_1

and $z=0$ is inside γ_1

we use Cauchy's Generalized Integral formula.

$$\int_{\gamma_1} \frac{e^z/z-i}{z^2}\, dz = \frac{2\pi i}{1!} \frac{d}{dz}\left(\frac{e^z}{z-i}\right)\Bigg|_{z=0}$$

$$= 2\pi i \; \frac{(z-i)e^z - e^z}{(z-1)^2}\Bigg|_{z=0} = 2\pi i\left(\frac{-i-1}{(-i)^2}\right)$$

$$= 2\pi i\,(1+i)$$

② $f(z) = e^z/z^2$ is analytic inside and on the contour γ_2 and $z=i$ is inside γ_2.

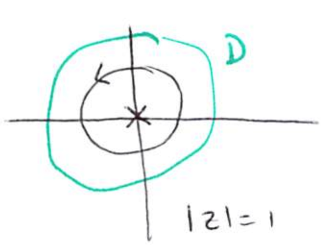
we use Cauchy's Integral Formula

$$\int_{\gamma_2} \frac{e^z/z^2}{z-i}\, dz = 2\pi i \; \frac{e^z}{z^2}\Bigg|_{z=i} = 2\pi i\left(\frac{e^i}{i^2}\right) = -2\pi i e^i$$

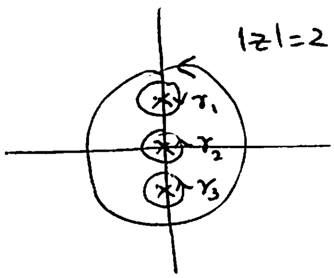
so $$\int_{|z|=2} \frac{e^z}{z^2(z-i)}\, dz = 2\pi i\,(1+i-e^i) = 2\pi i - 2\pi - 2\pi i e^i$$

(6) $\displaystyle\int_\gamma \frac{\sin z}{z(z^2+2)}\, dz$ singularities occur at $z=0$ and $z^2=-2$ or $z = \pm i\sqrt{2}$

$$\int_{|z|=1} \frac{\sin z}{z(z^2+2)}\, dz = \int_{|z|=1} \frac{\sin z/z^2+2}{z}\, dz = 2\pi i \; \frac{\sin z}{z^2+2}\Bigg|_{z=0} = 0$$



here we have used the Cauchy Integral Formula since $\sin z/_{z^2+2}$ is analytic in D (simply-connected) and z_0 is inside the curve $|z|=1$.



There are 3 singularities within the curve $|z| = 2$: $z_1 = i\sqrt{2}$, $z_2 = 0$, and $z_3 = -i\sqrt{2}$

We use deformation invariance to replace the curve $|z| = 2$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$.

we write

$$\int_{|z|=2} \frac{\sin z}{z(z+i\sqrt{2})(z-i\sqrt{2})}\, dz = \int_{\gamma_1} \frac{\frac{\sin z}{z(z+i\sqrt{2})}}{z-i\sqrt{2}}\, dz + \int_{\gamma_2} \frac{\sin z / z^2+2}{z}\, dz$$

$$+ \int_{\gamma_3} \frac{\sin z / z(z-i\sqrt{2})}{z+i\sqrt{2}}\, dz$$

and use Cauchy's integral formula for each of the three integrals

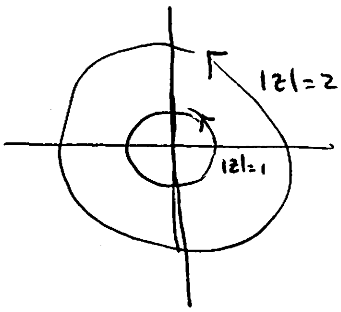
$$= 2\pi i \left.\frac{\sin z}{z(z+i\sqrt{2})}\right|_{z=i\sqrt{2}} + 2\pi i \left.\frac{\sin z}{z^2+2}\right|_{z=0} + 2\pi i \left.\frac{\sin z}{z(z-i\sqrt{2})}\right|_{z=-i\sqrt{2}}$$

$$= \frac{2\pi i \sin(i\sqrt{2})}{i\sqrt{2}(2i\sqrt{2})} + 0 + \frac{2\pi i \sin(-i\sqrt{2})}{-i\sqrt{2}(-2i\sqrt{2})}$$

$= 0$ since $\sin z$ is an odd function

(c)

$$\int_\Gamma \frac{\cosh z}{z^3}\,dz = \frac{2\pi i}{2!}\,\frac{d^2}{dz^2}(\cosh z)\Big|_{z=0} = \pi i \cosh z\Big|_{z=0} = \pi i$$



The only singularity is at $z=0$

$\cosh z$ is analytic everywhere in $\mathbb{C}$

we use the generalized Cauchy Integral formula

note that since both contours can be continuously transformed into one another the integral is the same for both curves.