Math 350, W2013

Assignment #3 - Solutions

#1. Give the complex form of the principal value of $e^{\sqrt{i}}$

First consider $\sqrt{i} = i^{1/2} = e^{1/2 \log i}$

with $\log i = \text{Log}\,|i| + i\,\text{Arg}\,i + 2k\pi i$ $\quad \to k = 0, 1$

$= \frac{i\pi}{2} + i2k\pi$

$k=0 \Rightarrow \sqrt{i} = e^{\frac{1}{2}(\frac{i\pi}{2})} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ 2

$k=1 \Rightarrow \sqrt{i} = e^{\frac{1}{2}(i\frac{\pi}{2} + i2\pi)} = e^{i\pi/4}e^{i\pi}$

$= e^{i5\pi/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

It follows that

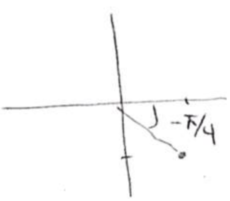
$e^{\sqrt{i}} = e^{1/\sqrt{2}}e^{i/\sqrt{2}} = e^{1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right)$ 1 /3

or $e^{\sqrt{i}} = e^{-1/\sqrt{2}}e^{-i/\sqrt{2}} = e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}}\right)$

#2 - Evaluate

(a) $(1-i)^{1+i}$ We use the formula $z^\alpha = e^{\alpha \log z}$

$(1-i)^{1+i} = e^{(1+i)\log(1-i)}$



$\log(1-i) = \text{Log}\,|1-i| + i\,\text{Arg}\,(1-i) + 2k\pi i$

$= \text{Log}\,2 - i\frac{\pi}{4} + 2k\pi i$ 1

and $(1+i)\log(1-i) = \text{Log}\,2 - \frac{i\pi}{4} + i2k\pi + i\log 2 + \frac{\pi}{4} - 2k\pi$

$= \text{Log}\,2 + \frac{\pi}{4} - 2k\pi + i\left(\text{Log}\,2 - \frac{\pi}{4} + 2k\pi\right)$

so $e^{(1+i)\log(1-i)} = e^{\text{Log}\,2 + \frac{\pi}{4} - 2k\pi}\, e^{i(\text{Log}\,2 - \frac{\pi}{4} + 2k\pi)}$ 1

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but $e^{i2k\pi} = 1$ and $e^{-i\pi/4} = \cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ 1

we can write $(1-i)^{1+i} = e^{\text{Log}2 + \frac{\pi}{4} - 2k\pi}\, e^{i(\text{Log}\,2 - \pi/4)}$ $\quad k \in \mathbb{Z}$ - note that there are ∞ many values

b) $\sinh(1+\pi i)$

we use the fact that $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$
with $z_1 = 1$ and $z_2 = \pi i$

$$\sinh(1+\pi i) = \sinh 1 \cosh \pi i + \cosh 1 \sinh \pi i \quad /$$

$$\sinh \pi i = \frac{e^{\pi i} - e^{-\pi i}}{2} = i\,\frac{e^{i\pi} - e^{-i\pi}}{2i} = i \sin \pi = 0$$

$$\cosh \pi i = \frac{e^{\pi i} + e^{-\pi i}}{2} = \cos \pi = -1 \quad /$$

so $\sinh(1+\pi i) = -\sinh 1$ $\quad /$ $\quad /3$

alternatively, $\sinh(1+\pi i) = \dfrac{e^{(1+\pi i)} - e^{-(1+\pi i)}}{2}$

$$= \frac{e^{1} e^{\pi i} - e^{-1} e^{-\pi i}}{2}$$

but $e^{\pi i} = \cos\pi + i\sin\pi = -1$
$e^{-\pi i} = \cos\pi - i\sin\pi = -1$

$$= -\frac{(e^{1} - e^{-1})}{2} = -\sinh 1 .$$

(c) i^{i^i}

we have $i^i = e^{i\log i} = e^{i(\mathrm{Log}|i| + i\pi/2)} = e^{-\pi/2}$ $\quad /$

so $i^{(i^i)} = e^{i^i \log i}$ where $i^i \log i = e^{-\pi/2}(\mathrm{Log}|i| + \frac{i\pi}{2})$

$$= e^{-\pi/2}\,\frac{i\pi}{2} \quad /$$

so the principal value of $i^{i^i} = e^{\frac{i\pi}{2}e^{-\pi/2}}$

$/ \quad /3$

3- Show that $\sin \bar{z}$ is nowhere analytic.

We need to write $\sin \bar{z}$ in the form $u + iv$, then use the Cauchy-Riemann equations.

Let $z = x + iy \Rightarrow \sin \bar{z} = \sin(x - iy)$

we use $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

with $z_1 = x$, $z_2 = iy$

so $\sin \bar{z} = \sin x \cos iy - \sin iy \cos x$

but $\cos iy = \dfrac{e^{i(iy)} + e^{-i(iy)}}{2} = \dfrac{e^{-y} + e^{y}}{2} = \cosh y$

$\sin iy = \dfrac{e^{i(iy)} - e^{-i(iy)}}{2i} = \dfrac{e^{-y} - e^{y}}{2i} = \dfrac{-(e^{y} - e^{-y})}{2i} =$

$= \dfrac{i(e^{y} - e^{-y})}{2} = i \sinh y.$

So $\sin \bar{z} = \sin x \cosh y - i \sinh y \cos x$ 2

$u(x,y) = \sin x \cosh y \Rightarrow u_x = \cos x \cosh y$

$u_y = \sin x \sinh y$ 5

$v(x,y) = -\sinh y \cos x \Rightarrow v_x = \sin x \sinh y$

$v_y = -\cos x \cosh y$

C.-R. (1) $u_x = v_y$

(2) $u_y = -v_x$

(1) $\cos x \cosh y = -\cos x \cosh y \iff \cos x = 0$ (note $\cosh y \neq 0$ for all y)

(2) $\sin x \sinh y = -\sin x \sinh y \iff$ either $\sin x = 0$

or $\sinh y = 0 \iff y = 0$

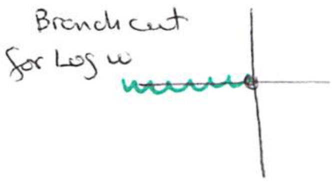
We conclude that $\sin \bar{z}$ is only differentiable at $y = 0$, $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

But it is nowhere analytic since these are isolated points, i.e., there are no open sets for which $\sin \bar{z}$ is differentiable.

#4 - Find the largest domain of analyticity for

(a) $f(z) = \mathrm{Log}(z^2)$

By definition, $\mathrm{Log}\, w$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$



or the set determined by $\mathrm{Re}\, w \leq 0$; $\mathrm{Im}\, w = 0$

Set $z^2 = -c$ for some $c > 0$, $c \in \mathbb{R}$

then $z = \pm\sqrt{-c} = \pm i\sqrt{c}$

Therefore any point on the imaginary axis corresponds to a point on the branch cut of $\mathrm{Log}(w)$

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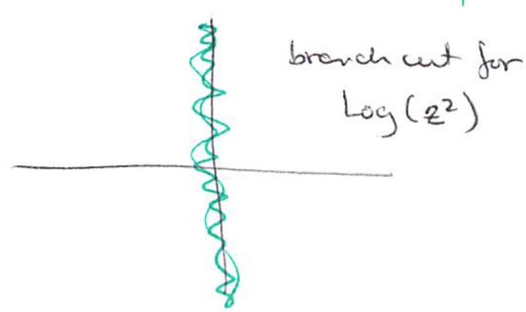
It follows that $\mathrm{Log}(z^2)$ is analytic on $\mathbb{C} \setminus \{\text{Im-axis}\}$



Alternatively, set $z = x + iy$

then $z^2 = (x+iy)^2 = -c$

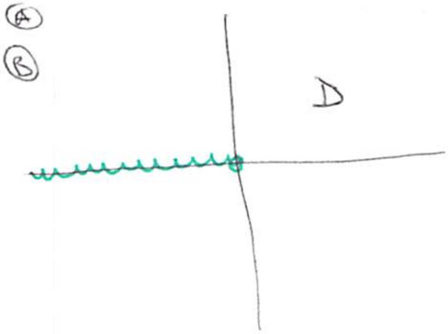
$x^2 + 2ixy - y^2 = -c$

or $x^2 - y^2 + i\,2xy = -c \Rightarrow x = 0$ and $y^2 = c$ or $y = \pm\sqrt{c}$

#4(b) $f(z) = \text{Log}(\text{Log}(z))$

Consider $\text{Log}\, w$ this function is analytic

in D where $D = \mathbb{C} - \begin{cases} \text{Im}\, w = 0 & (A) \\ \text{Re}\, w \leq 0 & (B) \end{cases}$



For $\text{Log}(\text{Log}\, z)$ we have

$w = \log z = \text{Log}\,|z| + i\, \text{Arg}\, z$

we need (A) $\text{Re}(w) = \text{Log}\,|z| \leq 0$

this is the "real" natural log

so $\log x \leq 0$ for $0 < x \leq 1$

So $0 < |z| \leq 1$ must be part of the branch cut

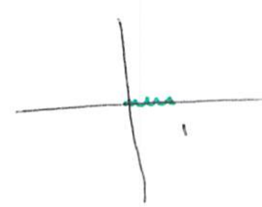
(B) we also require $\text{Im}\, w = \text{Arg}\, z = 0$

this means $z = x + iy$ $y = 0$ $x \geq 0$.

so z is pure real.

(A) + (B) Indicate that the points

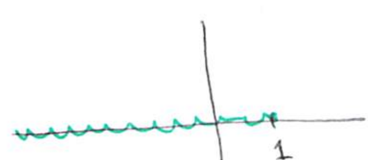
corresponding to $0 \leq x \leq 1$ $y = 0$ must be excluded.



But we also need $\text{Log}\, z$ to be

analytic so the usual branch cut must also be excluded.

The new Branch cut is



$D = \mathbb{C} \setminus \{ z \in \mathbb{C} \mid \text{Im}\, z = 0 \text{ and } \text{Re}\, w \leq 1 \}$

#5. Find all values of z for which $\cosh z = \frac{1}{2}$

we write $\cosh z = \dfrac{e^z + e^{-z}}{2}$

then $\cosh z = \dfrac{e^z + e^{-z}}{2} = \dfrac{1}{2} \iff e^z + e^{-z} - 1 = 0$

since $e^z \neq 0$

$\iff e^{2z} + 1 - e^z = 0$

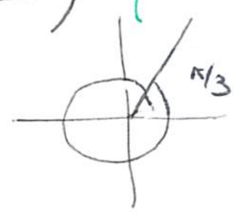
set $w = e^z$

$\iff w^2 - w + 1 = 0$

use the quadratic equation $w = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$

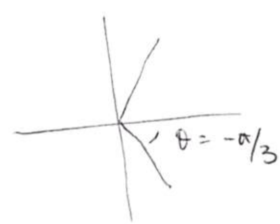
we need $e^z = \dfrac{1 \pm i\sqrt{3}}{2}$

or $z_1 = \log\left(\dfrac{1+i\sqrt{3}}{2}\right)$ and $z_2 = \log\left(\dfrac{1-i\sqrt{3}}{2}\right)$



$z_1 = \text{Log}\left|\dfrac{1+i\sqrt{3}}{2}\right| + i\,\text{Arg}\left(\dfrac{1+i\sqrt{3}}{2}\right) + i2k\pi \quad k \in \mathbb{Z}$

$= \text{Log}\,1 + \dfrac{i\pi}{3} + i2k\pi = i\left(\dfrac{\pi}{3} + 2k\pi\right)$



$z_2 = \text{Log}\left|\dfrac{1-i\sqrt{3}}{2}\right| + i\,\text{Arg}\left(\dfrac{1-i\sqrt{3}}{2}\right) + i2k\pi \quad k \in \mathbb{Z}$

$= -\dfrac{i\pi}{3} + i2k\pi$

$= i\left(-\dfrac{\pi}{3} + 2k\pi\right)$

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