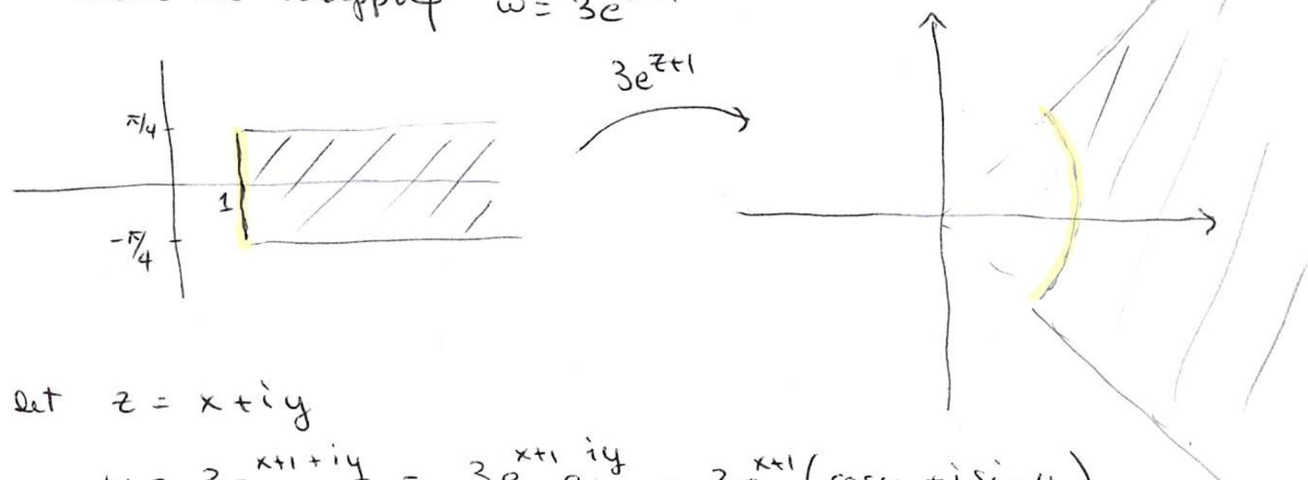
Math 350 W 2013

Test # 2 Solutions

#1 - Sketch the image of $\Omega = \{z \mid \operatorname{Re} z \geq 1, -\frac{\pi}{4} \leq \operatorname{Im} z \leq \frac{\pi}{4}\}$ under the mapping $w = 3e^{z+1}$



Let $z = x + iy$

$$w = 3e^{x+1+iy} = \underbrace{3e^{x+1}e^{iy}}_{Re^{i\theta}} = 3e^{x+1}(\cos y + i \sin y)$$

Fix y, then w is a ray with radius $3e^2 \leq R$

Fix x, Then w is the arc of a circle as θ sweeps from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

#2 - Find all complex values z such that

$$\sin z = \frac{i}{2}$$

$$\left(\frac{e^{iz} - e^{-iz}}{2i} = \frac{i}{2} \right) \times 2i$$

$$e^{iz} - e^{-iz} = -1$$

$$\left(e^{iz} - e^{-iz} + 1 = 0 \right) \times e^{iz}$$

$$e^{2iz} - 1 + e^{iz} = 0$$

Set $w = e^{iz}$ $\Rightarrow$ $w^2 + w - 1 = 0$

We need to find the roots of $w^2 + w - 1 = 0$

$$w = \frac{-1 \pm \sqrt{1-4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

next, we solve for $e^{iz} = \dfrac{-1 \pm \sqrt{5}}{2}$

Take log of both sides

$$i\left(iz = \operatorname{Log}\left|\tfrac{-1}{2} \pm \tfrac{\sqrt{5}}{2}\right| + i\operatorname{Arg}\left(\tfrac{-1}{2} \pm \tfrac{\sqrt{5}}{2}\right) + 2k\pi i \right) \qquad k \in \mathbb{Z}$$

$$-z = i\operatorname{Log}\left|\tfrac{-1}{2} \pm \tfrac{\sqrt{5}}{2}\right| - \operatorname{Arg}\left(\tfrac{-1}{2} \pm \tfrac{\sqrt{5}}{2}\right) - 2k\pi \qquad k \in \mathbb{Z}$$

$$z = -i\operatorname{Log}\left|\tfrac{-1}{2} \pm \tfrac{\sqrt{5}}{2}\right| + \operatorname{Arg}\left(\tfrac{-1}{2} \pm \tfrac{\sqrt{5}}{2}\right) + 2k\pi$$

note that $-1+\sqrt{5} > 0$ and $-1-\sqrt{5} < 0$

so $\operatorname{Arg}\left(\tfrac{-1}{2} + \tfrac{\sqrt{5}}{2}\right) = 0$ and $\operatorname{Arg}\left(\tfrac{-1}{\sqrt{2}} - \tfrac{\sqrt{5}}{2}\right) = \pi$

positive root gives

$$z_+ = -i\operatorname{Log}\left|\tfrac{-1}{2} + \tfrac{\sqrt{5}}{2}\right| + 2k\pi \qquad k \in \mathbb{Z}$$

negative root gives

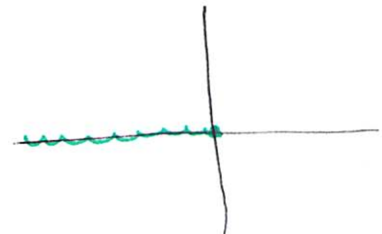
$$z_- = -i\operatorname{Log}\left|\tfrac{-1}{2} - \tfrac{\sqrt{5}}{2}\right| + \pi + 2k\pi$$

$$= -i\operatorname{Log}\left|\tfrac{-1}{2} - \tfrac{\sqrt{5}}{2}\right| + \pi(2k+1) \qquad k \in \mathbb{Z}$$

there are infinitely many values.

3 - (a) Sketch and describe the domain of analyticity of $f(z) = \mathrm{Log}(2i + z)$

Consider $f(w) = \mathrm{Log}(w)$ the domain of analyticity is



$\mathbb{C} \setminus$ the branch cut

which consist of all the points such that $\mathrm{Im}\, w = 0$ and $\mathrm{Re}\, w \le 0$

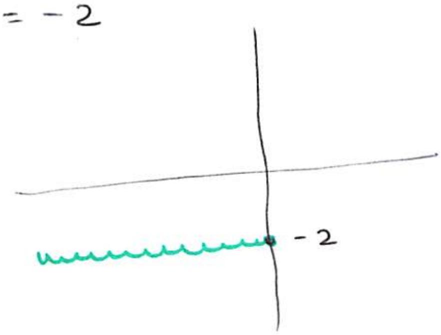
Here, $w = 2i + z$

Set $z = x + iy$ Then $w = 2i + x + iy$

$= x + i(2 + y)$

we need $2 + y = 0 \Rightarrow y = -2$

and $x \le 0$



so the domain of analyticity is the Complex Plane $\setminus$ the branch cut $\{z \mid x \le 0 \text{ and } y = -2\}$

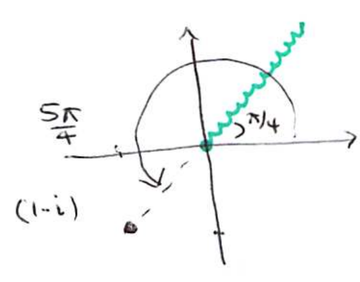
(b) Evaluate $\mathcal{L}_{\pi/4}(1 - i)$

By definition $\mathcal{L}_{\frac{\pi}{4}}(z) = \mathrm{Log}|z| + i\,\mathrm{Arg}_{\frac{\pi}{4}}(z)$ where

$\frac{\pi}{4} < \mathrm{Arg}_{\frac{\pi}{4}} \le \frac{\pi}{4} + 2\pi$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\mathrm{Arg}_{\frac{\pi}{4}}(1 - i) = \frac{5\pi}{4}$



so $\mathcal{L}_{\frac{\pi}{4}}(1 - i) = \mathrm{Log}\sqrt{2} + i\frac{5\pi}{4}$

$\frac{\pi}{4} < \mathrm{Arg}_{\frac{\pi}{4}} \le \frac{9\pi}{4}$

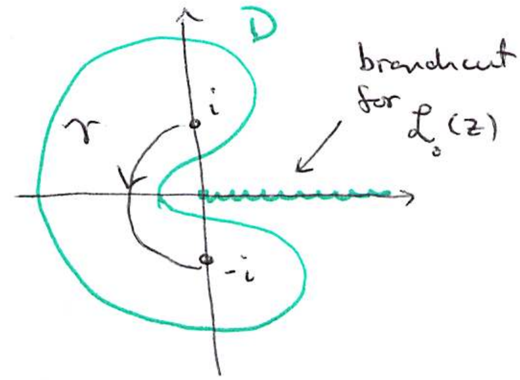
#4 (a) State the Fundamental theorem of Calculus for Contour

Let Γ be any contour in a domain D with endpoints z_0 and z_1. If $f: D \to \mathbb{C}$ is continuous and has an antiderivative $F(z)$ in D, then

$$\int_\Gamma f(z)\,dz = F(z_1) - F(z_0)$$

(b) Let γ be the part of the unit circle that joins the points $z = i$ to $z = -i$ traversed with positive orientation. Compute

$$\int_\gamma \frac{1}{z}\,dz$$



the antiderivative of $\frac{1}{z}$ is the log function but $\operatorname{Log} z$ is not differentiable on the branch cut so we need to use log so that D includes γ but no points on the branch cut.

Take $\mathcal{L}_0(z) = \operatorname{Log}|z| + i \operatorname{Arg}_0 z \qquad 0 < \operatorname{Arg}_0(z) \leq 2\pi$

then $\displaystyle\int_\gamma \frac{1}{z}\,dz = \mathcal{L}_0(z)\Big|_i^{-i} = \mathcal{L}_0(-i) - \mathcal{L}_0(i) = i\frac{3\pi}{2} - i\frac{\pi}{2}$

$= i\pi$

* we use the FTC since $\frac{1}{z}$ is continuous in D and $\frac{d}{dz}\frac{1}{z} = \mathcal{L}_0(z)$ in D

#5 (a) State Cauchy's Integral theorem

If f is analytic in a simply connected domain D and Γ is any loop in D, then

$$\oint_{\Gamma} f(z)dz = 0$$

(b) State The deformation Invariance Theorem

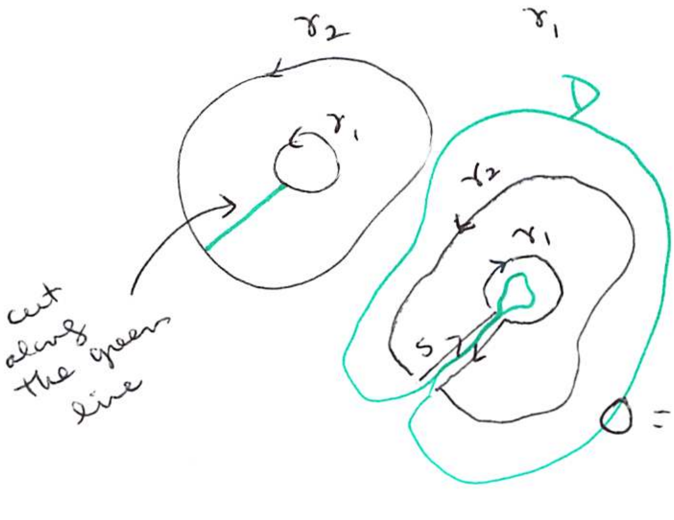
Let f be an analytic function in a domain D containing the loops γ_0 and γ_1. If these loops can be continuously deformed into one another in D then

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)\,dz$$

(c) We show (a) $\Rightarrow$ (b)

Assume f is analytic in a domain D, and that γ_1 and γ_2 are two loops in D that can be continuously deformed into one another. We need to show that

$$\oint_{\gamma_1} f(z)dz = \oint_{\Gamma_2} f(z)dz.$$



Consider The contour $\Gamma = \{\gamma_2, S, -\gamma_1, -S\}$

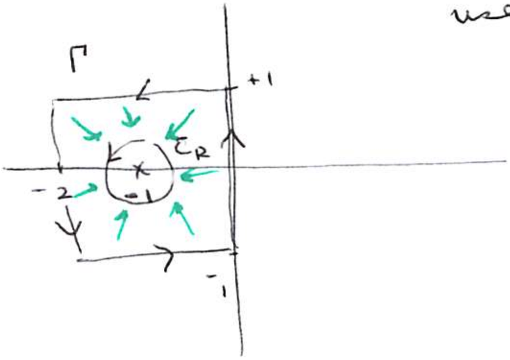
D is simply connected, f is analytic and Γ is a loop in D so by Cauchy's integral theorem

$$0 = \int_{\Gamma} f(z)dz = \int_{\gamma_2} f(z)dz + \int_{S} f(z)dz + \int_{-\gamma_1} f(z)dz + \int_{-S} f(z)dz$$

$$0 = \int_{\gamma_2} f(z)dz - \int_{\gamma_1} f(z)dz \iff \int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz$$

#6 (a) $$\int_\Gamma \frac{1}{1-z^2}\,dz = \int_\Gamma \frac{1}{(1-z)(1+z)}\,dz$$

use method of partial fraction



$$\frac{1}{(1-z)(1+z)} = \frac{A}{1-z} + \frac{B}{1+z} = \frac{A(1+z)+B(1-z)}{(1-z)(1+z)}$$

$$1 = A+B + z(A-B)$$

so $A+B=1$

$A = B = \frac{1}{2}$

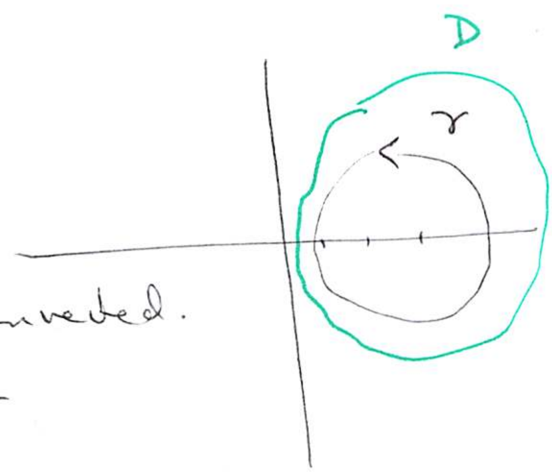
$$= \frac{1}{2}\int \frac{1}{1-z}\,dz + \frac{1}{2}\int \frac{1}{1+z}\,dz$$

$\frac{1}{1-z}$ is analytic everywhere inside and on the contour Γ
by Cauchy's integral theorem the first integral is 0.

$\frac{1}{1+z}$ has a singularity at $z=-1$, we use deformation invariance and the $2\pi i$ Lemma to obtain

$$= \frac{1}{2}\cdot 0 + \frac{1}{2}(2\pi i) = \pi i$$

(b) $$\int_\gamma \frac{e^z}{z}\,dz = 0$$



$\frac{e^z}{z}$ is analytic in D, D is simply connected.
by Cauchy's Integral theorem