Sample Test - Solutions

1. Sketch the curve given by the parametric equation

$$x = \ln t, \quad y = \sqrt{t}, \quad t \geq 1.$$

Your sketch should include the initial point, and the direction in which the curve is traced.

We can eliminate the parameter t by applying the exponential function to both sides of the equation:

$$e^x = e^{\ln t} = t \quad \Rightarrow \quad y = e^{x/2}.$$

The initial point corresponds to $t = 1 \Rightarrow \quad x = \ln 1 = 0, \; y = \sqrt{1} = 1$. As t increases, $x = \ln t$ increases and so does $y = e^{x/2}$.

2. Consider the curve $\mathcal{C}$ given by the equations $x = 2 - t^3$, $y = 2t - 1$, $z = \ln t$.

(a) Find a parametric equation for the tangent line to $\mathcal{C}$ at the point $(1, 1, 0)$.

The vector equation for the tangent line through the point $(1, 1, 0)$ is given by $(x, y, z) = (1, 1, 0) + t\vec{v}$, where $\vec{v}$ is the tangent vector to the curve at the point $(1, 1, 0)$. We have

$$\vec{r}(t) = (2 - t^3, 2t - 1, \ln t) \quad \text{and} \quad \vec{r}'(t) = (-3t^2, 2, \tfrac{1}{t}).$$

At $(1, 1, 0)$, $t = 1$, so $\vec{v} = \vec{r}'(1) = (-3, 2, 1)$.

The equation for the tangent line at $(1, 1, 0)$ in parametric form is

$$\begin{aligned} x(t) &= 1 - 3t, \\ y(t) &= 1 + 2t, \qquad\qquad t \in \mathbb{R}. \\ z(t) &= t, \end{aligned}$$

(b) Find an equation for the normal plane to $\mathcal{C}$ at the point $(1, 1, 0)$.

The normal plane at the point $(1, 1, 0)$ is the plane through $(1, 1, 0)$ with normal vector $\vec{n}$ parallel to the unit tangent vector $\vec{T} = \frac{\vec{r}'(t)}{||\vec{r}'(t)||}$. We pick $\vec{n} = (-3, 2, 1)$.

The equation for the plane is

$$\begin{aligned} \vec{n} \cdot (x - 1, y - 1, z - 0) &= 0 \\ (-3, 2, 1) \cdot (x - 1, y - 1, z) &= 0 \\ -3x + 2y + z + 3 - 2 &= 0 \\ 3x - 2y - z &= 1. \end{aligned}$$

3. Let $\mathcal{C}$ be a smooth plane curve.

(a) What is the osculating circle of $\mathcal{C}$ at the point p? What does it tell you about the curve?

The osculating circle at p is the circle containing the point p that best fits the curve at p. It lies in the osculating plane of the curve spanned by the unit tangent vector $\vec{T}$ and the principal unit normal vector $\vec{N}$.

The radius of the osculating circle ρ is related to the curvature κ of the curve by the equation $\rho = 1/\kappa$. Therefore, the osculating circle measures how 'curly' the curve is: the tighter the curve, the smaller the radius, the larger the curvature.

(b) Find an equation for the osculating circle of the curve $y = x^4 - x^2$ at the origin.

To find the radius of the osculating circle, we compute the curvature κ.
We use the fact that $y = x^4 - x^2$, $\quad y' = 4x^3 - 2x$, and $y'' = 12x^2 - 2$ in the formula for κ:

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|12x^2 - 2|}{[1 + (4x^3 - 2x)^2]^{3/2}}$$

At $(0, 0)$, $\kappa = | - 2| = 2$.
The osculating circle has radius $\rho = \frac{1}{2}$. At $(0, 0)$, $y' = 0$ and $y'' = -2$. Therefore the curve is concave down. This means that the osculating circle will be below the curve. The center of the osculating circle is therefore located at $(0, 0) - (0, \frac{1}{2}) = (0, -\frac{1}{2})$.
The equation for the osculating circle is given by

$$x^2 + (y + \tfrac{1}{2})^2 = \tfrac{1}{4}.$$

4. (a) State the Fundamental Theorem of Calculus for line integrals.

Let $\mathcal{C}$ be a smooth curve parametrized by $\vec{r}(t)$ for $a \le t \le b$. Let f be a differentiable function and let ∇f be continuous on $\mathcal{C}$. Then

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

(b) Prove the following statement:
"If $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ for every closed path $\mathcal{C}$ in D then $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ is path independent."

Proof: Assume $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ for any closed curve $\mathcal{C}$ in D. Let A, B be two points in D, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two paths in D joining A to B.
Consider the closed curve $\mathcal{C}$ obtained by traversing $\mathcal{C}_1$ followed by $-\mathcal{C}_2$. By assumption

$$\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} + \int_{-\mathcal{C}_2} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} \vec{f} \cdot d\vec{r} = 0.$$

It follows that

$$\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r}$$

Since $\mathcal{C}_1$ and $\mathcal{C}_2$ were arbitrary, we conclude that $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ is path independent.

5. (a) Find $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x + z)\vec{i} + z\vec{j} + y\vec{k}$ and $\mathcal{C}$ is the line from the point $(2, 4, 4)$ to the point $(1, 5, 2)$.

First, we parametrize the curve $\mathcal{C}$:

$$\begin{aligned} \vec{r}(t) =& (1 - t)(2, 4, 4) + t(1, 5, 2) \\ =& (2 - t, 4 + t, 4 - 2t), \qquad 0 \le t \le 1, \\ \vec{r}'(t) =& (-1, 1, -2). \end{aligned}$$

The integral is

$$\begin{aligned}\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} &= \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\;dt\\ &= \int_0^1 (2-t+4-2t, 4-2t, 4+t)\cdot(-1,1,-2)\;dt\\ &= \int_0^1 (-10-t)\;dt = \big(-10t-\tfrac{t^3}{2}\big)\Big|_0^1 = -\tfrac{21}{2}\end{aligned}$$

(b) Evaluate $\int_{\mathcal{C}}(3x-y)\;ds$, where $\mathcal{C}$ is the portion of the circle $x^2+y^2=18$ traversed from $(3,3)$ to $(3,-3)$ clockwise.

First we parametrize the curve $\mathcal{C}$:

$$\begin{aligned} x(t) &= \sqrt{18}\cos t, & y(t) &= -\sqrt{18}\sin t, & -\tfrac{\pi}{4}\le t\le \tfrac{\pi}{4}.\\ x'(t) &= -\sqrt{18}\sin t, & y'(t) &= -\sqrt{18}\cos t. \end{aligned}$$

This gives

$$ds = \sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2} = \sqrt{18\sin^2 t+18\cos^2 t} = \sqrt{18}.$$

We integrate

$$\begin{aligned}\int_{\mathcal{C}}(3x-y)\;ds &= \int_{-\pi/4}^{\pi/4}(3\sqrt{18}\cos t+\sqrt{18}\sin t)\sqrt{18}\;dt\\ &=18\int_{-\pi/4}^{\pi/4}(3\cos t+\sin t)\;dt\\ &=18(3\sin t-\cos t)\Big|_{-\pi/4}^{\pi/4}\\ &=18\left(3\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}+3\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}\right)=54\sqrt{2}.\end{aligned}$$

6. Let $\vec{F}(x,y)=(2xy+3)\vec{\imath}+(x^2+\cos y)\vec{\jmath}$

(a) Show that $\vec{F}$ is a conservative vector field.

We have $P(x,y)=(2xy+3)$, $Q(x,y)=(x^2+\cos y)$, and

$$\frac{\partial P}{\partial y}=2x=\frac{\partial Q}{\partial x}.$$

Also, $\vec{F}$ is defined for all $(x,y)\in\mathbb{R}^2$. So the domain is open and simply connected. It follows from the theorem that $\vec{F}$ is a conservative vector field.

(b) Find a potential function for $\vec{F}$.

We want $f(x,y)$ with $\nabla f(x,y)=(2xy+3,x^2+\cos y)$.

1. $f_x=2xy+3 \quad\Rightarrow\quad f(x,y)=\int(2xy+3)\;dx=x^2y+3x+g(y).$
2. $f_y=x^2+\cos y=x^2+g'(y) \quad\Rightarrow\quad g'(y)=\cos y \quad\Rightarrow\quad g(y)=\sin y+K.$

It follows that $f(x, y) = x^2y + 3x + \sin y + K$ is a potential function for $\vec{F}$.

(c) Use part (b) to compute $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ where $\mathcal{C}$ is the curve beginning at the point $(1, 0)$ and ending at the point $(2, \pi)$.

$$\int_{\mathcal{C}} \vec{F} \cdot \, d\vec{r} = f(2, \pi) - f(1, 0) = (4\pi + 6 + \sin \pi) - (0 + 3 + \sin 0) = 4\pi + 3.$$