Quiz #4 - Solutions

1. A particle starts at the point $(-2,0)$, moves along the $x-$axis to $(2,0)$, and then along the semi-circle $y=\sqrt{4-x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\vec{F}(x,y)=(x,x^3+3xy^2)$.

The curve $\mathcal{C}$ is a positively oriented, piecewise smooth, simple closed curve. The interior of the curve forms a region $D=\partial\mathcal{C}$, and $P(x,y)=x$ and $Q(x,y)=x^3+3xy^2$ are both continuously differentiable in the interior of D.

Set $\vec{\mathbf{F}}(x,y)=P(x,y)\vec{\mathbf{i}}+Q(x,y)\vec{\mathbf{j}}$, then

$$\frac{\partial P}{\partial y}=0, \qquad \frac{\partial Q}{\partial x}=3x^2+3y^2.$$

We can use Green's Theorem to compute the integral.

Set $x=r\cos\theta$ and $y=r\sin\theta$, then $D=\{0\le r\le 2,\ 0\le\theta\le\pi\ \}$.

$$W=\int_{\mathcal{C}}\vec{\mathbf{F}}\cdot d\vec{r}=\int\int_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\ dA=\int\int_D 3x^2+3y^2\ dA.$$

We use circular symmetry to express our curve and integrand in terms of polar coordinates.

$$W=\int_0^\pi\int_0^2 3r^2\ r\ dr\ d\theta=\int_0^\pi \frac{3r^4}{4}\Big|_0^2\ d\theta=\int_0^\pi 12\ d\theta=12\pi.$$

2. (a) Show that any vector field of the form $\mathbf{F}(x,y,z)=f(y,z)\,\mathbf{i}+g(x,z)\,\mathbf{j}+h(x,y)\,\mathbf{k}$ is incompressible.

A vector field is incompressible if $\text{div}\vec{F}=\nabla\cdot\vec{F}=0$.

$$\text{div }\vec{F}=\frac{\partial}{\partial x}f(y,z)+\frac{\partial}{\partial y}g(x,z)+\frac{\partial}{\partial z}h(x,y)=0$$

Therefore $\vec{F}$ is incompressible.

(b) Determine whether or not $\mathbf{F}(x,y,z)=ye^{-x}\,\mathbf{i}+e^{-x}\,\mathbf{j}+2z\,\mathbf{k}$ is conservative.

$\vec{F}$ is conservative if $\text{curl}\vec{F}=\vec{0}$.

$$\text{curl }\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ ye^{-x} & e^{-x} & 2z\end{vmatrix}=-2e^{-x}\vec{\mathbf{k}}\neq 0$$

$\vec{F}$ is not conservative.

(c) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G}=yz\,\mathbf{i}+xyz\,\mathbf{j}+xy\,\mathbf{k}$? Explain.

No. If there was such a vector field, then div(curl $\vec{G}$) $=0$. But,

$$\text{div }(yz\vec{\mathbf{i}}+xyz\vec{\mathbf{j}}+xy\vec{\mathbf{k}})=\frac{\partial}{\partial x}(yz)+\frac{\partial}{\partial y}(xyz)+\frac{\partial}{\partial z}(xy)=xz\neq 0.$$