Winter 2008 **Math 317**

Quiz #3

1. Find the work done by the force field $\vec{F}(x,u) = (x, y+2)$ in moving an object along an arch of the cycloid $\vec{r}(t) = (t - \sin t, 1 - \cos t)$, $0 \le t \le 2\pi$.

We have

$$W = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t \; dt,$$

where $\vec{F}(\vec{r}(t)) = (t - \sin t, 3 - \cos t)$ and $\vec{r}'(t) = (1 - \cos t, \sin t)$. We evaluate the intergral.

$$\begin{aligned}
W &= \int_0^{2\pi} (t - \sin t, 3 - \cos t) \cdot (1 - \cos t, \sin t) \; dt \\
&= \int_0^{2\pi} (t - t\cos t + \sin t \cos t - \sin t + 3 \sin t - \sin t \cos t) \; dy \\
&= \int_0^{2\pi} (t - t\cos t + 2 \sin t) \; dt \\
&= \frac{t^2}{2}\Big|_0^{2\pi} - \left(t \sin t\Big|_0^{2\pi} - \int_0^{2\pi} \sin t \; dt \right) + 2\cos t\Big|_0^{2\pi} \\
&= 2\pi^2 - 2\pi \sin(2\pi) + (\cos(2\pi) - 1) + 2(\cos(2\pi) - 1) \\
&= 2\pi^2.
\end{aligned}$$

2. Find the gradient vector field for $f(x,y,z) = x\cos(y/z)$.

$$\begin{aligned}
\nabla f(x,y,z) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\
&= \left(\cos\left(\frac{y}{z}\right), -x \sin\left(\frac{y}{z}\right) \cdot \frac{1}{z}, -x\sin\left(\frac{y}{z}\right) \cdot \frac{-y}{z^2} \right) \\
&= \left(\cos\left(\frac{y}{z}\right), -\frac{x}{z} \sin\left(\frac{y}{z}\right), \frac{xy}{z^2} \sin\left(\frac{y}{z}\right) \right).
\end{aligned}$$

3. Evaluate the line integral, where $\mathcal{C}$ is the right half of the circle $x^2 + y^2 = 16$.

$$\int_{\mathcal{C}} xy^4 \ ds$$

First, we parametrize the curve $\mathcal{C}$.

Set $x = 4\cos t$, $y = 4\sin t$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, then

$$\begin{aligned}
\vec{r}(t) &= (4\cos t, 4\sin t),\\
\vec{r}'(t) &= (-4\sin t, 4\cos t),\\
||\vec{r}'(t)|| &= \sqrt{16\sin^2 t + \cos^2 t} = 4.
\end{aligned}$$

We integrate

$$\begin{aligned}
\int_{\mathcal{C}} xy^4 \ ds &= \int_{-\pi/2}^{\pi/2} \vec{F}(\vec{r}(t))||\vec{r}'(t)|| \ dt\\
&+ \int_{-\pi/2}^{\pi/2} 4\cos t(4\sin t)^4 \ 4dt\\
&= 4^6 \int_{-\pi/2}^{\pi/2} \sin^4 t \cos t \ dt\\
&= 4^6 \int_{-1}^{1} u^4 \ du\\
&= 4^6 \ \frac{u^5}{5}\Big|_{-1}^{1} = \frac{2(4^6)}{5}.
\end{aligned}$$

Where we have used the following $u-$substitution: $u = \sin t$, $du = \cos t \ dt$.