Math 317 - Calculus IV

Quiz #2 - solutions

1. Find a vector function that represents the curve of intersection of the surfaces $z = \sqrt{x^2+y^2}$ and $z = 1 + y$.

Squaring both equations, we obtain

$$z^2 = x^2 + y^2 = 1 + 2y + y^2.$$

We solve for y in terms of the variable x, and use the result to solve for z in terms of x.

$$y = \frac{1}{2}(x^2-1) \qquad \Rightarrow \qquad z = 1 + y = 1 + \frac{1}{2}(x^2-1) = \frac{1}{2}(x^2+1).$$

We set $x = t$ to obtain the parametric representation for the curve of intersection:

$$x = t, \qquad y = \frac{1}{2}(t^2-1), \qquad z = \frac{1}{2}(t^2+1).$$

2. Reparametrize the curve $\mathbf{r}(t) = e^{2t}\cos 2t\ \mathbf{i} + 2\ \mathbf{j} + e^{2t}\sin 2t\ \mathbf{k}$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

Step 1: Compute $||\mathbf{r}'(t)||$.

$$\begin{aligned}\mathbf{r}(t) =& (e^{2t}\cos 2t, 2, e^{2t}\sin 2t),\\ \mathbf{r}'(t) =& (2e^{2t}\cos 2t + e^{2t}(-2\sin 2t), 0, 2e^{2t}\sin 2t + e^{2t}(2\cos 2t))\\ =& 2e^{2t}(\cos 2t - \sin 2t, 0, \sin 2t + \cos 2t).\\ ||\mathbf{r}'(t)|| =& \sqrt{4e^{4t}[(\cos 2t - \sin 2t)^2 + (\sin 2t + \cos 2t)^2]} = 2\sqrt{2}e^{2t}.\end{aligned}$$

Step 2: Evaluate the arc-length function $s(t)$.

$$s(t) = \int_0^t 2\sqrt{2}e^{2u}\ du = \tfrac{2\sqrt{2}}{2}e^{2u}\Big|_0^t = \sqrt{2}(e^{2t}-1).$$

Step 3: Solve for t in terms of s.

$$\tfrac{s}{\sqrt{2}} = e^{2t} - 1 \qquad \Rightarrow \qquad \ln\left(\tfrac{s}{\sqrt{2}} + 1 = e^{2t}\right) \quad \Rightarrow \quad \tfrac{1}{2}\ln\left(\tfrac{s}{\sqrt{2}}+1\right) = t.$$

Step 4: Rewrite $\mathbf{r}$ in terms of the parameter s

$$\mathbf{r}(s) = \left(\left(\tfrac{s}{\sqrt{2}}+1\right)\cos\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right), 2, \left(\tfrac{s}{\sqrt{2}}+1\right)\sin\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\right).$$

3. Find the unit tangent vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and the curvature $\kappa(t)$ for the curve given by $\mathbf{r}(t) = (2\sin t, 5t, 2\cos t)$.

We compute

$$\begin{aligned}
&\mathbf{r}(t) = (2\sin t, 5t, 2\cos t), \quad \mathbf{r}'(t) = (2\cos t, 5, -2\sin t),\\
&||\mathbf{r}'(t)|| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} = \sqrt{29}.\\
&\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \tfrac{1}{\sqrt{29}}(2\cos t, 5, -2\sin t), \quad \mathbf{T}'(t) = \tfrac{1}{\sqrt{29}}(-2\sin t, 0, -2\cos t),\\
&||\mathbf{T}'(t)|| = \tfrac{1}{\sqrt{29}}\sqrt{4\sin^2 t + 4\cos^2 t} = \tfrac{2}{\sqrt{29}}.\\
&\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \frac{\tfrac{1}{\sqrt{29}}(-2\sin t, 0, -2\cos t)}{\tfrac{2}{\sqrt{29}}} = -(\sin t, 0, \cos t).\\
&\kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} = \frac{\tfrac{2}{\sqrt{29}}}{\sqrt{29}} = \tfrac{2}{29}.
\end{aligned}$$