Winter 2009 **Math 317**

Quiz #1 - Solutions

1. Find a parametric equation for the path of a particle that moves along the circle $x^2+(y-1)^2=4$ three times around clockwise, starting at $(2,1)$.

This is the equation of a circle centered at $(0,1)$ with radius $r=2$. The following parametrization will do:

$$x = 2\cos t, \quad y = 1 - 2\sin t, \qquad 0 \le t \le 6.$$

2. Find the exact length of the curve given by $x = 1+3t^2$ and $y = 4+2t^3$ for $0 \le t \le 1$.

$$\int_0^1 \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(6t)^2+(6t^2)^2}\, dt = \int_0^1 6t\sqrt{1+t^2}\, dt$$

Let $u = 1+t^2$. $du = 2t\ dt$. Substitution yields

$$3\int_1^2 \sqrt{u}\ du = 3\left(\frac{2}{3}u^{3/2}\right)\Big|_1^2 = 2(2\sqrt{2}-1).$$

3. Show that the curve $x = \cos t$, $y = \sin t \cos t$, for $0 \le t \le 2\pi$, has two tangents at $(0,0)$ and find their equations. Sketch the curve.

First we compute the rate of change of y with respect to x.

$$\frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos^2 t - \sin^2 t = \cos 2t,$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos^2 t - \sin^2 t}{-\sin t} = -\frac{\cos 2t}{\sin t}.$$

Since $\cos t = 0$ at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, there is a self-intersection at the origin when $t = \frac{\pi}{2},\ \frac{3\pi}{2}$.

The slopes for the tangent lines at the origin are given by

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = -\frac{\cos\pi}{\sin\frac{\pi}{2}} = 1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{t=3\pi/2} = -\frac{\cos 3\pi}{\sin\frac{3\pi}{2}} = -1.$$

The equations for the tangent lines at the origin are $y = x$ and $y = -x$.

To obtain the graph, we use the information above, together with the fact that the curve has horizontal tangent lines whenever $\frac{dy}{dx} = 0$ ($t = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$), and vertical tangent lines whenever $\frac{dx}{dt} = 0$ ($t = 0,\ \pi$). Note that the curve also crosses the x-axis whenever $y = \sin t\cos t = 0$, which occurs at $t = 0,\ \pi,\ \frac{\pi}{2},\ \frac{3\pi}{2}$.

The graph of the curve is a Lissajous figure.