Midterm - solutions

1. Consider the curve with parametric equation

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad 0 \le t \le 4\pi.$$

(a) Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1-\cos t},$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{(1-\cos t)(\cos t) - \sin t(\sin t)}{(1-\cos t)^2}}{1-\cos t} = \frac{\cos t - 1}{(1-\cos t)^3} = \frac{-1}{(1-\cos t)^2}.$$

(b) At what points does the curve have **an horizontal tangent**?

We have an horizontal tangent whenever $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

$$\frac{dy}{dt} = \sin t = 0 \quad \text{at } \pm n\pi, \text{ so } 0,\ \pi,\ 2\pi,\ 3pi,\ 4\pi$$
$$\frac{dx}{dt} = 1 - \cos t = 0 \quad \text{at } 0,\ 2\pi,\ 4\pi.$$

we have horizontal tangents at $t = \pi$ and $t = 3\pi$. AT $t = 0,\ 2\pi$, and 4π, the rate of change $\frac{dy}{dx}$ is undefined so we expect to see kinks in the curve.

(c) Use part (a) and (b) to sketch the curve. Your sketch should include the initial point and the direction in which the curve is being traversed.

Since $-1 \le \cos t \le 0$, we have $0 \le 1 - \cos t \le 2$, so $y(t) \ge 0$ for all t. The minimum value, $y(t) = 0$, is obtained at $t = 0,\ 2\pi, 4\pi$, and the maximum value, $y(t) = 2$, is obtained at $t = \pi,\ 3\pi$. Since $\frac{dx}{dt} \ge 0$, $x(t)$ is increasing throughout the interval, and since $\frac{d^2y}{dx^2} < 0$, the curve is concave down. We can use this information to plot the graph of the curve (see class notes).

2. Consider the curve

$$\vec{r}(t) = (e^t \cos t, e^t \sin t, e^t), \qquad t \ge 0.$$

(a) Compute the curvature κ at the point $(1, 0, 1)$.

We use the fact that at $\vec{r}(0) = (1, 0, 1)$ and the formulas

$$\kappa(t) = \frac{||\vec{T}\,'(t)||}{||\vec{r}\,'(t)||}, \qquad \vec{T} = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}.$$

We have

$$\begin{aligned}
\vec{r}\,'(t) &= (e^t(\cos t - \sin t),\ e^t(\sin t + \cos t),\ e^t), \\
||\vec{r}\,'(t)|| &= e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1} \\
&= e^t\sqrt{\cos^2 t - 2\sin t\cos t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t + 1} \\
&= e^t\sqrt{3}.
\end{aligned}$$

$$\begin{aligned}
\vec{T} &= \frac{e^{-t}}{\sqrt{3}}\left(e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t\right) \\
&= \frac{1}{\sqrt{3}}(\cos t - \sin t,\ \sin t + \cos t,\ 1).
\end{aligned}$$

$$\begin{aligned}
\vec{T}\,'(t) &= \sqrt{3}\,(-\sin t - \cos t,\ \cos t - \sin t,\ 0) \\
||\vec{T}\,'(t)|| &= \frac{1}{\sqrt{3}}\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2} \\
&= \frac{1}{\sqrt{3}}\sqrt{\sin^2 t + 2\cos t \sin t + \cos^2 t + \cos^2 t - 2\sin t \cos t + \sin^2 t} = \frac{\sqrt{2}}{\sqrt{3}}.
\end{aligned}$$

It follows that

$$\kappa(t) = \frac{||\vec{T}\,'(t)||}{||\vec{r}\,'(t)||} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{e^{-t}}{\sqrt{3}} = \frac{\sqrt{2}}{3}e^{-t}, \qquad \text{at } t = 0, \quad \kappa(0) = \frac{\sqrt{2}}{3}.$$

(b) Write the equation for the normal plane at the point $(1, 0, 1)$.

The normal plane is the plane spanned by the vectors $\vec{B}(t)$ and $\vec{n}(t)$., so the normal to the plane is parallel to the tangent vector $\vec{T}(t)$. At $t = 0$, $\vec{T}(0) = \frac{1}{\sqrt{3}}(1,\ 1,\ 1)$ so we take $\vec{n} = (1,\ 1,\ 1)$. The equationfor the plane is given by

$$\begin{aligned}
&\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \\
&(1, 1, 1) \cdot (x - 1, y, z - 1) = 0 \\
&x + y + z - 2 = 0
\end{aligned}$$

(c) Reparametrize $\vec{r}(t)$ with respect to arclength measured from the point $(1, 0, 1)$ in the direction of increasing t.

First, we compute an expression for $s(t)$.

$$s(t) = \int_0^{\tau} ||\vec{r}\,'(t)||\ d\tau = \int_0^{\tau} \sqrt{3}e^{\tau}\ d\tau = \sqrt{3}e^{\tau}\ \Big|_0^t = \sqrt{3}(e^t - 1).$$

Solving for t yields

$$s = \sqrt{3}(e^t - 1) \quad \Rightarrow \quad \frac{s}{\sqrt{3}} + 1 = e^t \quad \Rightarrow \quad \ln\left(\frac{s}{\sqrt{3}} + 1\right) = t.$$

We substitute this expression into $\vec{r}(t)$ to get the reparametrization by arclength.

$$\vec{r}(s(t)) = \vec{r}(s) = \left(\left(\frac{s}{\sqrt{3}} + 1\right) \cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \left(\frac{s}{\sqrt{3}} + 1\right) \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \frac{s}{\sqrt{3}} + 1 \right).$$

3. Compute the following line integral where $\mathcal{C}$ is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation.

$$\int_{\mathcal{C}} -y\ dx + (x + y^2)\ dy$$

First we need to parametrize $\mathcal{C}$. We take

$$x(t) = 3\cos t, \quad y(t) = 2\sin t, \quad \text{for } 0 \le t \le 2\pi,$$

then

$$dx = -3\sin t\ dt, \qquad \text{and} \qquad dy = 2\cos t\ dt.$$

We evaluate the line integral

$$\begin{aligned}
\int_{\mathcal{C}} -y\ dx + (x+y^2)\ dy &= \int_0^{2\pi} (-2\sin t)(-3\sin t)\ dt + \int_0^{2\pi} (3\cos t + 4\sin^2 t)(2\cos t)\ dt \\
&= \int_0^{2\pi} (6\sin^2 t + 6\cos^2 t + 8\cos t \sin^2 t)\ dt \\
&= \int_0^{2\pi} 6\ dt = 6t\ \Big|_0^{2\pi} = 12\pi.
\end{aligned}$$

4. (a) State the Fundamental Theorem of Calculus for line integrals.

Let $\mathcal{C}$ be a smooth curve given by $\vec{r}(t)$ for $a \le t \le b$. Let f be a differentiable function of 2 or 3 variables such that ∇f is continuous on $\mathcal{C}$. Then

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

(b) Let $\vec{F}(x,y) = (e^x + y^2 e^{xy})\ \vec{\mathbf{i}} + (1+xy)e^{xy}\ \vec{\mathbf{j}}$. Show that $\vec{F}$ is conservative.

We use theorem 6, 17.3:

1. $\vec{F}$ is defined for any $(x,y) \in \mathbb{R}^2$, so the region $D = \mathbb{R}^2$ is open and simply connected.
2. We set $P(x,y) = e^x + y^2 e^{xy}$ and $Q(x,y) = (1+xy)e^{xy}$, then

$$\frac{\partial P}{\partial x} = e^x + y^3 e6xy, \qquad \frac{\partial P}{\partial y} = 2ye^{xy} + y^2 x e^{xy},$$
$$\frac{\partial Q}{\partial x} = ye6xy + (1+xy)ye^{xy}, \qquad \frac{\partial Q}{\partial y} = xe^{xy} + (1+xy)xr^{xy}.$$

   so P and Q have continuous first order partial derivatives.
3. $\frac{\partial P}{\partial y} = 2ye^{xy} + y^2 x e^{xy} = \frac{\partial Q}{\partial x}$.

It follows from theorem 6, 17.3, that $\vec{F}$ is conservative.

(c) **Use part (a)** to evaluate $\int_{\mathcal{C}} \vec{F} \cdot\ d\vec{r}$ where $\mathcal{C}$ is the line segment from $(1,2)$ to $(4,0)$.

We need to find the potential function $f(x,y)$ such that $\vec{F} = \nabla f$. This means that we need

(a) $f_x = P(x,y) = e^x + y^2 e^{xy}$.
(b) $f_y = Q(x,y) = (1+xy)e^{xy}$.

By (a),

$$f(x,y) = \int P(x,y)\ dx = \int (e^x + y^2 e^{xy})\ dx = e^x + ye^{xy} + g(y).$$

It follows that,

$$f_y = e^{xy} + yxe^{xy} + g'(y) = (1+xy)e^{xy} + g'(y).$$

By (b) $f_y = Q(x,y)$ so $g'(y) = 0$, or $g(y) = k$ for some constant k. It follows that the potential function for $\vec{F}$ is $f(x,y) = e^x + ye^{xy} + k$.

We use part (a) to evaluate the integral

$$\int_{\mathcal{C}} \vec{f} \cdot \, d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(4,0) - f(1,2) = e^4 - e - 2e^2.$$

5. Find the work done by the force field $\vec{F}(x,y,z) = z\ \vec{\mathbf{i}} + x\ \vec{\mathbf{j}} + y\ \vec{\mathbf{k}}$ in moving a particle from the point $(3,0,0)$ to the point $(0, \frac{\pi}{2}, 3)$ along a straight line.

First we parametrize the line from $(3,0,0)$ to $(0, \frac{\pi}{2}, 3)$.

$$\begin{aligned}
&\vec{r}(t) = (1-t)(3,0,0) + t(0, \frac{\pi}{2}, 3) = (3-3t, \frac{\pi}{2}t, 3t), \qquad 0 \le t \le 1, \\
&\vec{r}\,'(t) = (-3, \frac{\pi}{2}, 3).
\end{aligned}$$

We evaluate the integral

$$\begin{aligned}
W &= \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} \\
&= \int_0^1 (3t, 3-3t, \frac{\pi}{2}t) \cdot (-3, \frac{\pi}{2}, 3)\ dt \\
&= \int_0^1 (-9t + \frac{\pi}{2})\ dt \\
&= \left(\frac{-9t^2}{2} + \frac{3\pi}{2}t\right) \Big|_0^1 \\
&= -\frac{9}{2} + \frac{3\pi}{2}.
\end{aligned}$$