Math 160 **Fall 2008**

Assignment # 5- Solutions

#1. True or False? Justify your answers.

(a) If a number is divisible by 8, then it is divisible by 2 and 4.

True, if a natural number n is divisible by 8, then $n = 8 \cdot m$ for some natural number m. But this means that we can write $n = 2 \cdot 4 \cdot m$. So n is divisible by 2 and by 4.

(b) If a number is divisible by 2 and 3, then it is divisible by 6.

True, since both 2 and 3 are prime, they must appear in the prime factorization of n. So $n = 2 \cdot 3 \cdot l = 6 \cdot l$ for some integer l. So n is divisible by 6.

(c) If a number is divisible by 2 and 4, then it is divisible by 8.

False. 12 is divisible by 2 and by 4, but not by 8.

#2. Use the Euclidean algorithm to find the GCD(1739, 5291). Show your work.

$$\begin{array}{r|l} 1739 & 5291 \\ \hline & 5217 \\ \hline & 74 \end{array}$$

$$\begin{array}{r|l} 74 & 1739 \\ \hline & 1702 \\ \hline & 37 \end{array}$$

$$\begin{array}{r|l} 37 & 74 \\ \hline & 74 \\ \hline & 0 \end{array}$$

$\text{GCD}(1739, 5291) = \text{GCD}(74, 1739) = \text{GCD}(37, 74) = 37.$

#3. If $\text{GCD}(X, 4389) = 33$ and $\text{LCM}(1739, 5291) = 566,181$, what is X?

We use the fact that $a \cdot b = \text{GCD}(a, b) \cdot \text{LCM}(a, b)$ with $a = X$ and $b = 4389$. Solving for X yields

$$x = \frac{33 \cdot 566181}{4389} = 4257.$$

#4. Use the four-step process to answer the following questions.

(a) Find the greatest 4-digit number that has exactly three factors.

step 1: Understand the problem

We look for the largest 4-digit natural number n whose set of divisors contains only 3 numbers. That is $D_n = \{a, b, c\}$. Note that since 1 and n are part of this set, $D_n = \{1, b, n\}$. So we need to identify the number b.

step 2: Devise a plan

I will use the fact that every natural number has a unique prime factorization:

$$n = p_1^{a_1} p_2^{a_2} \cdots {}_r^{a_r},$$

and the theorem that states that the number of divisors for n is exactly

$$d = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1).$$

step 3: Carry out the plan

We need $d = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1) = 3$. This can only happen if $a_1 = 2$ and all other exponents are zero. The prime factorization for n is as follows:

$$n = p^2$$

We need to find the largest prime whose square is a 4-digit number. Because $100^2 = 10,000$ is a 5-digit number, we need a prime less than 100. We try the closest prime number. Since $97^2 = 9409$, the number we are looking for is $n = 9409$.

step 4: Looking back

The set of divisor $D_9 409 = \{1, 97, 9409\}$.

(b) Find the least number divisible by each of the natural number less than or equal to 12.

step 1: Understand the problem

I need to find the smallest natural number m that is divisible by each of the terms in the set $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12\ \}$. But this means I need the least common multiple.

step 2: Devise a plan

I will find the prime factorization of m. That is, the prime factorization of n is made up of the smallest set of primes that includes the prime factorization of each of the numbers in the set S.

step 3: Carry out the plan

First, I write the prime factorization for the numbers in S that are not prime.

prime numbers: 2, 3, 5, 7, 11

$4 = 2 \cdot 2$

$6 = 2 \cdot 3$

$8 = 2 \cdot 2 \cdot 2$

$9 = 3 \cdot 3$

$10 = 2 \cdot 5$

$12 = 2 \cdot 2 \cdot 3$

The least common multiple is given by

$$m = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 27,720.$$

step 4: Looking back

We check that m is divisible by each of the numbers in S.

$27,720 \div 1 = 27,720$

$27,720 \div 2 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 13,860$

$27,720 \div 3 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 9240$

$27,720 \div 4 = \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 6930$

$27,720 \div 5 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 \cdot 11 = 5544$

$27,720 \div 6 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 4620$

$27,720 \div 7 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 11 = 3960$

$27,720 \div 8 = 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 3465$

$27,720 \div 9 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7 \cdot 11 = 3080$

$27,720 \div 10 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 \cdot 11 = 2772$

$27,720 \div 11 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 2520$

$27,720 \div 12 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$