integrate with respect to y .

$$\frac{2}{a}\int_0^a F(x,y)\sin \lambda_s x\, dx = \sum_m A_{m,s} \sin \gamma_m y \cancel{\tfrac{a}{2}}$$

$$\frac{2}{a}\int_0^b\int_0^a F(x,y)\sin\lambda_s x \sin\gamma_r y\, dx\,dy = \int_0^b \sum_m A_{m,s}\sin\gamma_m y \sin\gamma_r y\, dy$$

$$= \sum_m A_{m,s}\int_0^b \sin\gamma_m y \sin\gamma_r y\, dy$$

$$= A_{r,s}\,\frac{b}{2}$$

so $\Rightarrow$

$$A_{m,n} = \frac{2}{b}\times\frac{2}{a}\int_0^b\int_0^a F(x,y)\sin\lambda_n x \sin\gamma_m y\, dx\,dy .$$

10.6 - the Wave Equation

recall. $\Rightarrow$ 1-dim^l wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

c depends on the physical parameters of the string .

$u(0,t) = u(L,t) = 0 \qquad t \geq 0$

$u(x,0) = f(x) \quad , \quad \frac{\partial u}{\partial t}(x,0) = g(x)$

$\rightarrow$ For any value of n ~~$u_n(x,t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$~~

each can be viewed as a standing wave.

$$u_n(x,t) = \underbrace{\left(a_n \sin\frac{n\pi c t}{L} + b_n\cos\frac{n\pi c t}{L}\right)}_{\text{Time varying amplitude}} \underbrace{\sin\frac{n\pi x}{L}}_{\text{sinusoidal curve}}$$

is a solution

These are called the normal modes

the eigenfrequencies of a string / drum, . . . are

the harmonics $\Rightarrow \omega_n = \dfrac{n\pi c}{L}$

$\to$ fundamental — is the frequency of the lowest mode $\dfrac{\pi c}{L}$

harmonics — integer multiple of the fundamental.

The solution is expressed as a superposition of infinitely many standing waves. $\boxed{\text{(finite domain)}}$

Now, ∞ domain:

So . PDE $\longrightarrow$ $u_{tt} - c^2 u_{xx} = 0$

Consider $u(x,t) = F(x+ct) + G(x-ct)$ where F & G are twice differentiable funct. on $(-\infty, \infty)$

$$u_t = F'(x+ct)\cdot c + G'(x-ct)\cdot(-c)$$

$$u_{tt} = F''(x+ct)\,c^2 + G''(x-ct)\cdot c^2$$

$$u_x = F'(x+ct) + G'(x-ct)$$

$$u_{xx} = F''(x+ct) + G''(x-ct)$$

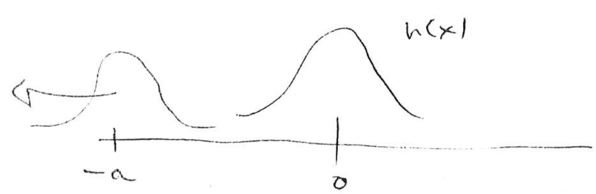
so

$$u_{tt} - c^2 u_{xx} = c^2(F'' + G'') - c^2(F'' + G'') = 0$$

$u(x,t)$ satisfies our P.D.E .

Do EX 2 first (2 paragraphs)

* Traveling Waves



Consider $h(x)$

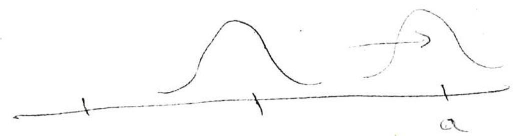
$h(x+a)$ $a>0$ gives a translation

the shape has been shifted to the left by an amount a.

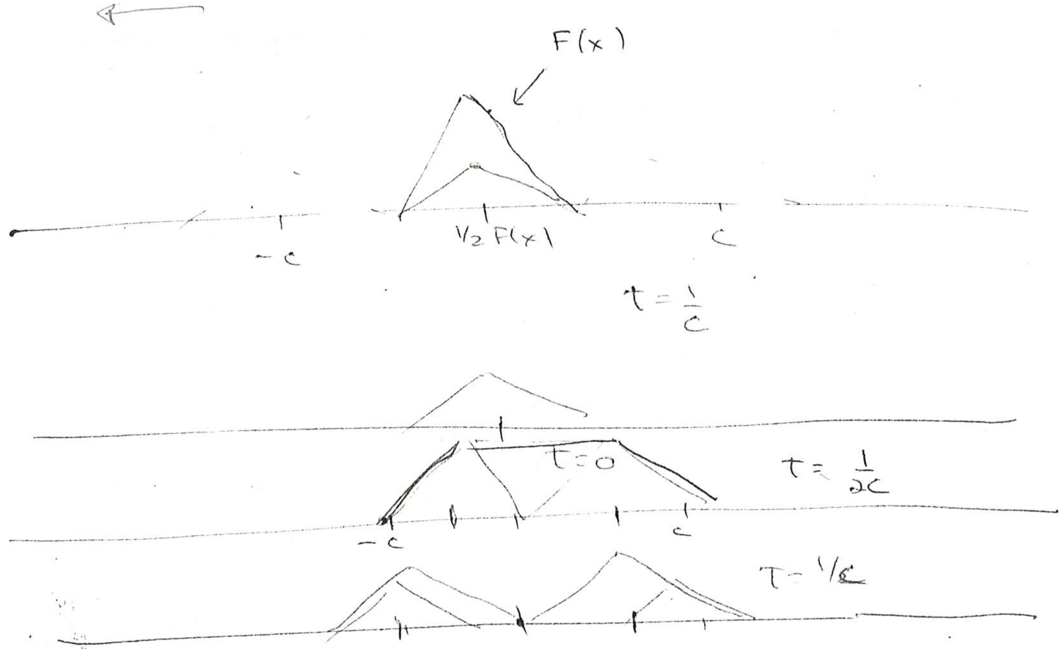
Let $t \geq 0$ be a parameter

consider $h(x+at)$ this is a family of function with the same shape but ~~st~~ moving to the left as $t \to \infty$.

Likewise $h(x-at)$ —
is a family of curve
with the shape moving to the right
as $t \to \infty$



$$u(x,t) = \frac{1}{2}\left[F(x+ct) + G(x-ct)\right]$$



as t increase, the two waves move away from each other with speed $2c$.

Ex. 1. Consider an ∞-strip that is initially perturbed from rest and gently released.

$$u_{tt} - c^2 u_{xx} = 0$$ this describes the vertical motion of the strip

$u(x,0) = \varphi(x)$ $\quad -\infty < x < \infty$ initial position

$u_t(0,x) = 0$ $\quad -\infty < x < \infty$ released from rest.

Then $u(x,t) = F(x+ct) + G(x-ct)$ is a general solution

Check. $u(x,0) = F(x) + G(x) = \varphi(x)$

$u_t(x,0) = cF'(x) - cG'(x) = 0$

So we want $F'(x) = G'(x)$

& $F(x) + G(x) = \varphi(x)$

Set $F = G = \frac{1}{2}\varphi$

then $u(x,t) = \left[\frac{1}{2}(\varphi(x+ct) + \varphi(x-ct))\right]$

These are known as the d'Alembert Solution

Jean Le Rond d'Alembert

French scholar - published in 1747

it was discovered independently and only slightly later by Euler.

Day 9 Oct 6

9.1

Ex. 2 $u_{tt} - c^2 u_{xx} = 0 \qquad -\infty < x < \infty$

$t > 0$

wave "acoustic equation"

$u(x,0) = \varphi(x) \qquad -\infty < x < \infty$

$u_t(x,0) = \xi(x) \qquad -\infty < x < \infty$

soundwaves

magneticwaves

assume $\quad u(x,t) = F(x+ct) + G(x-ct)$

Apply BC $\Rightarrow u(x,0) = F(x) + G(x) = \varphi(x)$

$u_t(x,0) = c\left(F'(x) - G'(x)\right) = \xi(x) \longleftarrow$ integrate

$$F(x) - G(x) = \frac{1}{c}\int_{x_0}^{x} \xi(s)\,ds + K \qquad \ldots(1)$$

x_0 - arbitrary

K, c - constant,

but also

$$F(x) + G(x) = \varphi(x) \qquad (2)$$

① (1)+(2) $\quad 2F(x) = \varphi(x) + \frac{1}{c}\int_{x_0} \xi(s)\,ds + K$

$$\Rightarrow F(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_{x_0}^{x} \xi(s)\,ds + \frac{K}{2}$$

(1)-(2)

② $\quad 2G(x) = \varphi(x) - \frac{1}{c}\int_{x_0}^{x} \xi(s)\,ds - K$

$$\Rightarrow G(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_{x_0}^{x} \xi(s)\,ds - \frac{K}{2}$$

so $\quad u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\left[\int_{x_0}^{x+ct} \xi(s)\,ds - \int_{x_0}^{x-ct} \xi(s)\,ds\right]$

$$= \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \xi(s)\,ds$$

$\boxed{\text{finite translation speed!}}$

(skip')

* In text p.616.

Turns out we can obtain d'Alembert solution by using a method similar to what we used to solve 1st-order PDE.

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

"family of lines".

set $\psi = x + \alpha t$, $\eta = x - \alpha t$.

think of u as a function of ψ & η

if u has continuous 2nd partial derivative

then. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \psi} + \frac{\partial u}{\partial \eta}$ i.e $\frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$

likewise $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = \alpha\frac{\partial u}{\partial \psi} - \alpha\frac{\partial u}{\partial \eta}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \psi}\left(\frac{\partial u}{\partial \psi}\right)\frac{\partial \psi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \psi}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \psi}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \psi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial^2 u}{\partial \psi^2} + 2\frac{\partial^2 u}{\partial \eta \partial \psi} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \alpha\frac{\partial}{\partial \psi}\left(\frac{\partial u}{\partial \psi}\right)\frac{\partial \psi}{\partial t} + \alpha\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \psi}\right)\frac{\partial \eta}{\partial t} - \alpha\frac{\partial}{\partial \psi}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \psi}{\partial t} - \alpha\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial t}$$

$$= \alpha^2\frac{\partial^2 u}{\partial \psi^2} - \alpha^2\frac{\partial^2 u}{\partial \eta \partial \psi} - \alpha^2\frac{\partial^2 u}{\partial \psi \partial \eta} + \alpha^2\frac{\partial^2 u}{\partial \eta^2} = \alpha^2\left\{\frac{\partial^2 u}{\partial \psi^2} - 2\frac{\partial^2 u}{\partial \eta \partial \psi} + \frac{\partial^2 u}{\partial \eta^2}\right\}$$

Substitute back in DE

$$\Rightarrow \frac{\partial^2 u}{\partial \eta \partial \psi} = 0$$

① Integrate with respect to η — an arbitrary function of ψ

or $\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \psi}\right) = 0 \Rightarrow \frac{\partial u}{\partial \psi} = b(\psi)$

② Integrate with respect to ψ

$$u = \int b(\psi)\,d\psi + c(\eta)$$

$$= F(\psi) + G(\eta)$$

in terms of x, t)

$$u(x,t) = F(x+\alpha t) + G(x-\alpha t)$$

RECAP!

Heat Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad 0<x<L$$

* need Spatial B.C. homogeneous

B.C. in x $\qquad t>0$

Dirichlet $\quad u(0,t) = u(L,t) = 0$

Neuman $\quad u_x(0,t) = u_x(L,t) = 0$

Mixed Type $\quad u(0,t) = u_x(L,t) = 0$

$\quad u_x(0,t) = u(L,t) = 0$

Robin $\quad c_1 u(0,t) + c_2 u_x(0,t) = 0$

and $\quad u(L,t)$ or $u_x(L,t) = 0$

or the reverse.

$F_n(x) = a_n \sin\lambda_n x + b_n \cos\lambda_n x$ $\quad \boxed{\lambda_0 = 0,\ F_0(x) = 1 \text{ could have}}$

Remaing condition

initial temperature

$u(x,0) = f(x) \qquad 0<x<a$.

Non-homogeneous B.C or non-homogeneous PDE.

set $u(x,t) = v(x) + w(x,t)$

$v(x)$ ↙ steady state; $w(x,t)$ ↘ Transient solution

set up $v(x)$ so that $w(x,t)$ has hom. B.C. and hom PDE

Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \qquad 0<x<L$$

* need spatial BC. homogeneous

$u(x,t) = F(x)G(t)$

general solution are

$F_n(x) = a_n \sin\lambda_n x + b_n \cos\lambda_n x$

$G_n(t) = c_n \sin\lambda_n a t + d_n \cos\lambda_n a t$

* Non-homogeneous Problem

* series solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{L}$$

Ex. Non-hom. Problem

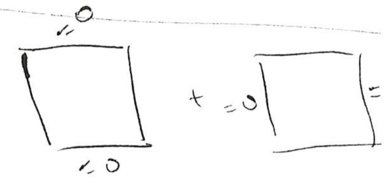
$u(x,y) = u_p + w$

$u_p = u_1(x) + u_2(y)$

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0<x<a,\ 0<y<b$$

* need at least one set of spatial B.C. homogeneous.



If B.C. are homogeneous in x

$F(x) = A\cos\lambda_n x + B\sin\lambda_n x$

$G(y) = A\cosh\lambda_n y + B\sinh\lambda_n y$

$= A\cosh\lambda_n y + B\sinh\lambda_n(y-$

$= A\sinh\lambda_n y + B\cosh\lambda_n(y-b)$

depending on whether the B.C. are given as $u(x,0)=0$ or $u_y(x,0)=$

$u(x,b)=0$ or $u_y(x,b)=$

If at least one of the B.C. is hom. in y use it to eliminate one of the constant ⟸

otherwise

$F(y) = A\sinh\lambda_n y + B\sinh\lambda_n(y$

If B.C are homogeneous in y.

$G(y) = A\cos\lambda_n y + B\sin\lambda_n y$

$F(x)$ ....

linear combination of sinh & cosh ..

Dirichlet $\quad \phi'' + \lambda^2\phi = 0$, $\ \phi(0) = \phi(a) = 0 \Rightarrow \lambda_n = \frac{n\pi}{a}$, $\ \phi_n = \sin\frac{n\pi x}{a}$

Neuman $\quad \phi'' + \lambda^2\phi = 0$, $\ \phi'(0) = \phi'(a) = 0 \Rightarrow \lambda_0 = 0 \Rightarrow \phi_0(x) = 1$

$\lambda_n = \frac{n\pi}{a} \qquad n = 1, 2, \ldots$

$\phi_n(x) = \cos\frac{n\pi x}{a}$

Chapter 11

We have been solving PDEs:

I If the problem is not homogeneous, split it up so that you have non-homogeneous + homogeneous bits.

To solve the homogeneous PDE:

① apply sep^n of vars

② set up the ODE problems

③ solve the ODE problems

    (A) BVP

    (B) other (t or y-dep.)

④ put the solutions from (A) + (B) together to form the sol'n. ①

⑤ apply the remaining IC or BC

⑥ find the Fourier coefficients.

Which parts require the most work?

③ (A) The BVP!!!

In this section, we just focus on the BVP, & we look @ more complex BVPs.

The situation for the nonhomogeneous problem (1)–(2) is similar. However, there is an additional possibility: *The nonhomogeneous problem may have no solutions.* The following examples illustrate the various situations.

Example 1 Find all the solutions to the boundary value problem

(5) $$y'' + 2y' + 5y = 0 \;;$$

(6) $$y(0) = 2 \,, \qquad y(\pi/4) = 1 \;.$$

Solution The auxiliary equation for (5) is $r^2 + 2r + 5 = 0$, which has roots $r = -1 \pm 2i$. Hence, a general solution for (5) is $y(x) = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$. We will now try to determine c_1 and c_2 so that the boundary conditions in (6) are satisfied. Setting $x = 0$ and $\pi/4$, we find

$$y(0) = c_1 = 2 \,, \qquad y(\pi/4) = c_2 e^{-\pi/4} = 1 \;.$$

Hence, $c_1 = 2$ and $c_2 = e^{\pi/4}$, and so problem (5)–(6) has the unique solution

$$y(x) = 2e^{-x} \cos 2x + e^{\pi/4} e^{-x} \sin 2x \;. \quad ◆$$

Example 2 Find all the solutions to the boundary value problem

(7) $$y'' + y = \cos 2x \;;$$

(8) $$y'(0) = 0 \,, \qquad y'(\pi) = 0 \;.$$

Solution The auxiliary equation for (7) is $r^2 + 1 = 0$, so a general solution for the corresponding homogeneous equation is $y_h(x) = c_1 \cos x + c_2 \sin x$. As we know from the method of undetermined coefficients, a particular solution to (7) has the form $y_p(x) = A \cos 2x + B \sin 2x$. Substituting y_p into (7) and solving for A and B, we find $A = -1/3$ and $B = 0$. Hence, $y_p(x) = -(1/3) \cos 2x$. Therefore, a general solution for (7) is

(9) $$y(x) = c_1 \cos x + c_2 \sin x - (1/3) \cos 2x \;.$$

To determine the values c_1 and c_2, we substitute the general solution in (9) into (8) and obtain

$$y'(0) = c_2 = 0 \,, \qquad y'(\pi) = -c_2 = 0 \;.$$

Thus, $c_2 = 0$ and c_1 is arbitrary. So problem (7)–(8) has a one-parameter family of solutions, namely,

$$y(x) = c_1 \cos x - (1/3) \cos 2x \,,$$

where c_1 is any real number. ◆

Example 3 Find the solutions to

(10) $$y'' + 4y = 0 \;;$$

(11) $$y(-\pi) = y(\pi) \,, \qquad y'(-\pi) = y'(\pi) \;.$$

Solution The auxiliary equation for (10) is $r^2 + 4 = 0$, so a general solution is

(12) $$y(x) = c_1 \cos 2x + c_2 \sin 2x \;.$$

Since $\cos 2x$ and $\sin 2x$ are π- (and hence 2π-) periodic functions, it follows that $y(-\pi) = y(\pi)$. Moreover, $y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$ is also π-periodic and so $y'(-\pi) = y'(\pi)$. Thus, (10)–(11) has a two-parameter family of solutions given by (12), where c_1 and c_2 are the parameters. ◆

Let's see what happens if, in the preceding example, we replace equation (10) by the nonhomogeneous equation

$$y'' + 4y = 4x \ . \tag{13}$$

A general solution for (13) is

$$y(x) = x + c_1 \cos 2x + c_2 \sin 2x \ . \tag{14}$$

Since $y(-\pi) = -\pi + c_1$ and $y(\pi) = \pi + c_1$, we find that there are no solutions to (13) that satisfy $y(-\pi) = y(\pi)$. Thus, the nonhomogeneous boundary value problem (13)–(11) has *no solutions.*

The Sturm–Liouville boundary value problems introduced in the preceding section are examples of two-point boundary problems that involve a parameter λ. Our goal is to determine for which values of λ the boundary value problem

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda r(x)y = 0 \ , \qquad a < x < b \ , \tag{15}$$

$$a_1 y(a) + a_2 y'(a) = 0 \ , \qquad b_1 y(b) + b_2 y'(b) = 0 \ , \tag{16}$$

has nontrivial solutions. Such problems are called **eigenvalue problems.** The nontrivial solutions are called **eigenfunctions** and the corresponding number λ an **eigenvalue.**†

One reason eigenvalue problems are important is that they arise in using the method of separation of variables to solve partial differential equations (see Section 10.2). It is therefore worthwhile to keep in mind some simple examples of these problems that were solved in Chapter 10.

Dirichlet boundary conditions:

$$y'' + \lambda y = 0 \ ; \qquad y(0) = y(L) = 0 \ , \tag{17}$$

which has eigenvalues $\lambda_n = (n\pi/L)^2$, $n = 1, 2, 3, \ldots$, and corresponding eigenfunctions

$$\phi_n(x) = a_n \sin\left(\frac{n\pi x}{L}\right) \ ,$$

where the a_n's are arbitrary nonzero constants (see Section 10.2).

Neumann boundary conditions:

$$y'' + \lambda y = 0 \ ; \qquad y'(0) = y'(L) = 0 \ , \tag{18}$$

†Readers familiar with linear algebra will recognize these eigenvalues and eigenfunctions as the eigenvalues and eigenvectors for the linear operator defined by $\mathscr{L}[y] := -(1/r)(py')' - (q/r)y$, where the domain of $\mathscr{L}$ consists of those twice-differentiable functions satisfying (16).

Chapt 11

Eigenvalue Problems . .

Find a solution to

DE: $y'' + p(x)y' + q(x)y = f(x)$ $\quad a < x < b$

satisfying B.C.

BC: $a_{11}y(a) + a_{12}y'(a) + b_{11}y(b) + b_{12}y'(b) = c_1$
$a_{21}y(a) + a_{22}y'(a) + b_{21}y(b) + b_{22}y'(b) = c_2$

} linear B.C.

if $c_1 = c_2 = 0$

* Homogeneous

$\boxed{\text{* TWO-POINT BOUNDARY VALUE PROBLEM}}$

their are certain B.C. that occur frequently in application

Dirichlet $\quad y(a) = c_1 \;, y(b) = c_2$

Neuman $\quad y'(a) = c_1 \;, \; y'(b) = c_2$

Periodic $\quad y(-T) = y(T) \;, \; y'(-T) = y'(T)$

or $\; y(0) = y(2T) \;, \; y'(0) = y'(2T)$ if period is $2T$

Separated $\quad a_1 y(a) + a_2 y'(a) = c_1$
$b_1 y(b) + b_2 y'(b) = c_2$

if neither a_1 & a_2 are non-zero then we have Robin B.C.

if $a_1 = 0$, $b_2 = 0$ we have mixed

If the DE & Boundary condition are homogeneous . .

Either we have a unique solution, a one parameter family of solution or a two-parameter family of solution. (case 1, 2, 3) } homogeneous case

~~only one linearly independent solution~~ ~~$y(x) = A\phi_1(x)$~~
~~two~~ ~~$y(x) = A\phi_1(x) + B\phi_2(x)$~~

~~or a unique sol. $y(x) = 0$~~

nonhomogeneous case: may have NO SOL'N

* this leads to an eigenvalue problem. ⟹ General case
General

Ex. Consider the Heat Equation

$x=a$ $x=b$

proportional to temperature

indep of temperature

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left[k(x)\frac{\partial u}{\partial x}(x,t)\right] + \underbrace{q(x)u(x,t) + h(x,t)}_{\text{internal source/sink}}$$

$c(x)$ = specific heat capacity of a material

$\rho(x)$ = density of the thin wire

$k(x)$ - thermal conductivity of the material

with B.C

$$\alpha_1 u(a,t) + \alpha_2 \frac{\partial u}{\partial x}(a,t) = 0$$

$$\beta_1 u(b,t) + \beta_2 \frac{\partial u}{\partial x}(b,t) = 0$$

* Robin type Radiating Heat

Look at the case when $h(x,t)=0$ - (homogeneous P.D.E)

separate variable.

$$u(x,t) = F(x)G(t)$$

$$c(x)\rho(x)F(x)G'(t) = [k(x)F'(x)]'G(t) + q(x)F(x)G(t)$$

$$\frac{1}{FG}\left[c(x)\rho(x)F(x)G'(t) = k(x)F''(x)G(t) + k'(x)F'(x)G(t) + q(x)F(x)G(t)\right]$$

$$c(x)\rho(x)\frac{G'(t)}{G(t)} = k(x)\frac{F''(x)}{F(x)} + \frac{k'(x)F'(x)}{F(x)} + q(x)$$

$$\frac{G'(t)}{G(t)} = \frac{k(x)}{c(x)\rho(x)}\frac{F''(x)}{F(x)} + \frac{k'(x)}{c(x)\rho(x)}\frac{F'(x)}{F(x)} + \frac{q(x)}{c(x)\rho(x)}$$

$$\frac{G'}{G} = \underbrace{\frac{1}{c(x)\rho(x)}\left[\frac{[k(x)F'(x)]' + F(x)q(x)}{F(x)}\right]}_{} = -\lambda$$

Sturm-Liouville Problem ⟹

$$[k(x)F'(x)]' + q(x)F(x) + \lambda c(x)\rho(x)F(x) = 0$$

① $\alpha_1 F(a) + \alpha_2 F'(a) = 0$

② $\beta_1 F(b) + \beta_2 F'(b) = 0$

11.3 ~~§6.2~~ - Sturm-Liouville Theory.

Shain covered 11.1 - 11.3 up to Eigenfunction Expansion

A regular Sturm-Liouville problem is a boundary-value problem on a closed interval $[a,b]$ of the form

Sturm-Liouville form

ODE (1) $[p(x)y']' + [q(x) + \lambda r(x)]y = 0$ $\quad a < x < b,$

B.C. (2) (a) $c_1 y(a) + c_2 y'(a) = 0$
(b) $d_1 y(b) + d_2 y'(b) = 0$ $\}$ separated B.C.

where at least one of c_1, c_2 is non-zero
and one of d_1, d_2 is non-zero.
λ is a parameter.

that satisfies

regularity condition $\{$ $p(x)$, $p'(x)$, $q(x)$, and $r(x)$ are real-valued continuous on $a \le x \le b$
with $p(x) > 0$, and $r(x) > 0$ on $a \le x \le b$.

* Singular Sturm-Liouville problems occur when:
finite interval with one of the conditions are false
or infinite interval.

Ex. of a regular S-L problem.

$y'' + \lambda y = 0$
$y(0) = y(\pi) = 0$

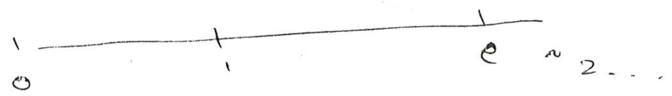
$\{ p = 1, \; q = 0, \; r = 1, \; c_2, d_2 = 0$

or $y'' + \lambda y = 0$
$y'(0) = y'(\pi) = 0$

Ex. 2. $(xy')' + \frac{\lambda}{x} y = 0$ $\quad 1 < x < e$

$y'(1) = 0, \quad y(e) = 0$

* similar to
Ex 5. p 652.
Dirichlet.



on this interval

$p(x) = x \qquad p'(x) = 1$

$q(x) = 0 \qquad r(x) = \frac{1}{x}$

p & r are continuous.
$p, r > 0$

so this is a regular Sturm-Liouville problem.

$\Rightarrow \quad xy'' + y' + \frac{\lambda}{x} y = 0$

Cauchy type Equation

$x^2 y'' + xy' + \lambda y = 0.$

Set $\quad y = x^r$
$y' = r x^{r-1}$
$y'' = r(r-1)x^{r-2}$

Then $\quad r(r-1)x^r + r x^r + \lambda x^r = 0$

$x^r(r^2 - r + r + \lambda) = 0$

$r^2 + \lambda = 0$

$r^2 = -\lambda$

We need to consider 3 cases $\quad \lambda = 0, \quad \lambda < 0 \; (\lambda = -\mu^2, \mu > 0)$
$\lambda > 0 \; (\lambda = \mu^2, \mu > 0)$

$\boxed{\lambda = 0}$ in original ODE

$\Rightarrow (xy')' = 0$

$xy' = c_1$

$y' = \frac{c_1}{x}$

$y = c_1 \ln x + c_2$

apply B.C.

$y'(1) = 0 = c_1$

$y(e) = 0 = c_2$

only get a trivial solution

$\lambda = 0$ is not an eigenvalue.

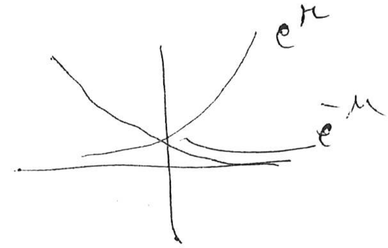
$\boxed{\lambda < 0}$ set $\lambda = -\mu^2 \; ; \; \mu > 0$

characteristic equation yield $r^2 = \mu^2$

$r = \pm\mu$

$y(x) = c_1 x^{\mu} + c_2 x^{-\mu}$

$y'(x) = c_1 \mu x^{\mu-1} - c_2 \mu x^{-\mu-1}$



$y'(1) = 0 = \mu(c_1 - c_2) \Rightarrow c_1 = c_2$

so $y(x) = c_1(x^{\mu} + x^{-\mu})$

$y(e) = 0 = c_1 (e^{\mu} + e^{-\mu})$ but $e^{\mu} = -e^{-\mu}$ only when $\mu = 0$ ↯

So $c_1 = c_2 = 0$

only get trivial solution

$\boxed{\lambda > 0}$ Set $\lambda = \mu^2 \; ; \; \mu > 0$

$r^2 = -\mu^2$ so $r = \pm i\mu$ roots are complex.

---

* if $r = \alpha \pm i\beta \Rightarrow$ consider $x^{\alpha + i\beta}$

Euler formula. $x^{\alpha + i\beta} = e^{(\alpha + i\beta)\ln x} = e^{\ln x^{(\alpha + i\beta)}}$

$= e^{\alpha \ln x} e^{i\beta \ln x}$

$= x^{\alpha} [\cos(\beta \ln x) + i \sin(\beta \ln x)]$

this yields 2 linearly independent solution

$y_1 = x^{\alpha} \cos(\beta \ln x)$

$y_2 = x^{\alpha} \sin(\beta \ln x)$

here $r = \pm i\mu$ is pure imaginary.

so the general solution is

$$y(x) = c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)$$

$$y'(x) = -c_1 \frac{\mu}{x} \sin(\mu \ln x) + c_2 \frac{\mu}{x} \cos(\mu \ln x)$$

$$y'(1) = 0 = c_2 \mu \quad \Rightarrow \quad c_2 = 0$$

so $y(x) = c_1 \cos(\mu \ln x)$

$$y(e) = c_1 \cos(\mu \ln e) = c_1 \cos \mu = 0$$

$$\Rightarrow \quad \mu = \frac{2n-1}{2}\pi$$

So $\lambda_n = \left[\left(\frac{2n-1}{2}\right)\pi\right]^2 \qquad n = 1, 2, \ldots$

$$y_n(x) = \cos\left(\left(\frac{2n-1}{2}\right)\pi \ln x\right)$$

these are eigen functions. So $y(x) = \sum_{n=1}^{\infty} c_n \cos\left(\left(\frac{2n-1}{2}\right)\pi \ln(x)\right)$.

* we will see in §11.3 - that if $\phi_n(x)$ are eigenfunctions for a Regular Sturm Liouville problem that we can write

$$f(x) = \sum c_n \phi_n(x)$$ $\Leftarrow$ Eigenfunction (Representation) Expansion

($\because$ the eigenfns of Reg S-L problems are $\perp$)

EIGENFUNCTION EXPANSIONS

Theory: Suppose we have an orthoNORMAL set of eigenfunctions $\varphi(x)$ wrt a weight fn $w(x)$ on $[a, b]$. Then

$$\int_a^b \varphi_n(x)\varphi_m(x) w(x)\,dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n, \end{cases}$$

and for ANY piecewise continuous fn $f(x)$, we can identify an orthogonal expansion $f(x) \sim \sum_{n=1}^{\infty} c_n \varphi_n(x)$ where $c_n = \int_a^b f(x)\varphi_n(x) w(x)\,dx$.

$\Rightarrow$ conversion

$$\frac{1}{x}\left(x^2y'' - \lambda x y' + \lambda y = 0\right)$$

$$xy'' - \lambda y' + \lambda y = 0$$

$$xy'' + y' - y' - \lambda y' + \lambda y = 0$$

$$(xy')' = xy'' + y'$$

$$\Rightarrow \quad (xy')' - (\lambda+1)y' + \frac{\lambda}{x}y = 0$$

so $\quad p(x) = x$

It turns out that the eigenfns of a Reg S-L problem are orthogonal. So we can use these to write fns f in eigenfn expansions.

a) The Reg S-L problem gives us a set of orthogonal eigenfunctions

$$y_n(x) = \cos\left(\frac{(2n-1)}{2}\pi \ln(x)\right).$$

To make an orthoNormal set of eigenfns, we normalize ...

A Regular Sturm-Liouville BVP

$$(p(x)y'(x))' + q(x)y(x) + \lambda r(x)y(x) = 0, \quad a < x < b,$$

with

$$\begin{cases} a_1 y(a) + a_2 y'(a) = 0 \\ b_1 y(b) + b_2 y'(b) = 0 \end{cases}$$

where

$p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are real-valued continuous functions on $[a,b]$ and, $p(x) > 0$ & $r(x) > 0$ on $[a,b]$. Also, at least one of a_1 & a_2 is nonzero, & at least one of b_1 & b_2 is nonzero.

- Thm 2: The eigenvalues λ for the regular Sturm-Liouville BVP are real & have real-valued eigenfunctions.

- Thm 4: Eigenfunctions that correspond to distinct eigenvalues of the regular Sturm-Liouville BVP are orthogonal with respect to the weight function $r(x)$ on $[a,b]$.

$\therefore$ The eigenfunctions of a regular Sturm-Liouville BVP are orthogonal, and these orthogonal eigenfunctions can be used to express any function $f(x)$ as an eigenfunction expansion on $[a,b]$.

(5)

Nov 4th Lindsey Reinholtz

4. $y''-2y+\lambda y=0$ $\quad 0<x<\pi$

$y(0)=0$

$y'(\pi)=0$

a) $(y')'-2y+\lambda y=0$. ← Sturm-Liouville Equation.

In order for the boundary value problem

$(p(x)y(x))'+q(x)y(x)+\lambda r(x)y(x)=0 \qquad a<x<b$

to be regular, we require that

① $p(x), p'(x), q(x)$ and $r(x)$ are real-valued continuous functions on $[a,b]$.

② $p(x)>0$, $r(x)>0$ on $[a,b]$.

Our boundary value problem is regular as

- $p(x)=1$ is real-valued and continuous on $0<x<\pi$.
- $p'(x)=0$ is real-valued and continuous on $0<x<\pi$.
- $q(x)=-2$ is real-valued and continuous on $0<x<\pi$
- $r(x)=1$ is real-valued and continuous on $0<x<\pi$.
- $p(x)=1>0$ and $r(x)=1>0$ on $[0,\pi]$.

b) $y''-2y+\lambda y=0$

Set $y=e^{rx}$

$y'=re^{rx}$

$y''=r^2e^{rx}$

$r^2e^{rx}-2e^{rx}+\lambda e^{rx}=0$

$r^2=2-\lambda$

$r=\pm\sqrt{2-\lambda}$

Consider 3 Cases:

<u>Case 1:</u> $\lambda=2$

$y''=0$

$y'=A$

$y=Ax+B$.

$y(0) = A(0) + B = 0$
$B = 0$
$\therefore y(x) = Ax$

$y'(\pi) = A = 0$

Only get the trivial solution.

<u>Case 2:</u> $\lambda < 2 \rightarrow 2-\lambda > 0$. say $\mu^2 = 2-\lambda$, so $r = \pm\sqrt{\mu^2} = \pm\mu$

$y(x) = c_1 \sinh \mu x + c_2 \cosh \mu x$
$y'(x) = c_1 \mu \cosh \mu x + c_2 \mu \sinh \mu x$

$y(0) = c_1(0) + c_2(1) = 0$
$c_2 = 0.$

$\therefore y(x) = c_1 \sinh \mu x.$

$y'(\pi) = c_1 \mu \cosh(\mu\pi) = 0$
$c_1 = 0.$

Only get the Trivial Solution.

<u>Case 3:</u> $\lambda > 2 \rightarrow 0 > 2-\lambda.$ say $-\mu^2 = 2-\lambda$, so $r = \pm\sqrt{2-\lambda} = \pm i\mu.$

$y(x) = c_3 \cos \mu x + c_4 \sin \mu x$
$y'(x) = -c_3 \mu \sin \mu x + c_4 \mu \cos \mu x$

$y(0) = c_3(1) + c_4(0) = 0$
$c_3 = 0.$

$\therefore y(x) = c_4 \sin \mu x.$

$y'(\pi) = c_4 \mu \cos \mu\pi = 0$

$c_4 = 0$ $\qquad$ $\cos \mu\pi = 0$

Trivial Solution

(6)

Nov 4th

Lindsey Reinholz

$\cos \mu\pi = 0$

$\mu\pi = \left(\frac{2n-1}{2}\right)\pi$ for $n=1,2,\ldots$

$\mu = (2n-1)/2 \rightarrow -\mu^2 = -(2n-1)^2/4 \rightarrow 2-\lambda_n = -(2n-1)^2/4$

$\lambda_n = 2+(2n-1)^2/4$

Eigenvalues: $\lambda_n = 2+(2n-1)^2/4$ for $n=1,2,\ldots$

Eigenfunctions: $y_n(x) = c_n \sin \mu_n x = c_n \sin\left(\frac{2n-1}{2}\right)x$

To normalize the eigenfunctions, we must have

$$\int_a^b y_n(x)^2 r(x)\,dx = 1$$

In this case $r(x)=1$ with $0<x<\pi$.

Consider $\lambda_n = \mu_n^2 = n^2$ with $y_n(x) = c_n \cos nx$.

$$\int_0^{\pi} c_n^2 \sin^2 \mu_n x \,dx = 1$$

$$c_n^2 \int_0^{\pi} \left(\frac{1-\cos 2\mu_n x}{2}\right) dx = 1$$

$$\frac{c_n^2}{2}\left(x - \frac{\sin(2\mu_n x)}{2\mu_n}\right)\Bigg|_0^{\pi} = 1$$

$$\frac{c_n^2}{2}\left[\left(\pi - \frac{\sin(2\mu_n\pi)}{2\mu_n}\right) - 0\right] = 1$$

Since $\mu_n = \frac{2n-1}{2}$, then $2\mu_n = 2n-1 \in \mathbb{Z}$, so $\sin(2\mu_n\pi)=0$,

$$\frac{c_n^2}{2}\pi = 1$$

$$c_n = \sqrt{\frac{2}{\pi}}$$

The normalized eigenfunction is:

$$\boxed{y_n(x) = \sqrt{2/\pi}\,\sin\left(\frac{2n-1}{2}\right)x}$$

c) $f(x) = x$

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x).$$

where $a_n = \int_0^{\pi} f(x) y_n(x) r(x)\, dx$.

Compute a_n:

$$a_n = \int_0^{\pi} x \sqrt{\frac{2}{\pi}} \sin \mu_n x \, dx$$

Integration by Parts:

$u = x \qquad dv = \sin \mu_n x \, dx$

$du = dx \qquad v = -\frac{\cos \mu_n x}{\mu_n}$

$$a_n = \sqrt{\frac{2}{\pi}} \left[\frac{-x \cos \mu_n x}{\mu_n} \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos \mu_n x}{\mu_n} dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\underbrace{\frac{-\pi \cos \mu_n \pi}{\mu_n}}_{0} - 0 + \frac{\sin \mu_n x}{\mu_n^2} \Big|_0^{\pi} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \mu_n \pi}{\mu_n^2} - 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin \frac{2n-1}{2}\pi}{\left(\frac{2n-1}{2}\right)^2} = \left(\frac{2}{2n-1}\right)^2 \sqrt{\frac{2}{\pi}} (-1)^{n+1}$$

$$f(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{2}{2n-1}\right)^2 (-1)^{n+1} \sqrt{\frac{2}{\pi}} \sin\left(\frac{2n-1}{2} x\right)$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{2}{2n-1}\right)^2 (-1)^{n+1} \sin\left(\frac{2n-1}{2} x\right)}$$

Comparing the new formula to the one from Ch 10

~~Let f(x) = 0 ≤ x ≤ π/2~~

$$\text{Let } f(x) = \begin{cases} \dfrac{2x}{\pi} & 0 \le x \le \dfrac{\pi}{2} \\ 1 & \dfrac{\pi}{2} \le x \le \pi \end{cases}$$

Before:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)}{2} x\right)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin\left(\frac{(2n-1)}{2} x\right) dx$$

So
$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\int_0^{\pi} f(x) \sin\left(\frac{(2n-1)}{2} x\right) dx\right] \sin\left(\frac{(2n-1)}{2} x\right)$$

Now:

$$f(x) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{\pi}} \sin\left(\frac{(2n-1)}{2} x\right)$$

$$c_n = \int_0^{\pi} f(x) \sqrt{\frac{2}{\pi}} \sin\left(\frac{(2n-1)}{2} x\right) dx = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x) \sin\left(\frac{(2n-1)}{2} x\right) dx$$

So
$$f(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \left[\int_0^{\pi} f(x) \sin\left(\frac{(2n-1)}{2} x\right) dx\right] \sqrt{\frac{2}{\pi}} \sin\left(\frac{(2n-1)}{2} x\right)$$

$\frac{2}{\pi}$ as before ☺

General Theory: Converting a 2nd order ODE into a Sturm-Liouville Eqn

* Given

$$A_2(x)\, y''(x) + A_1(x) y'(x) + A_0(x) y(x) + \lambda \rho(x) y(x) = 0 \quad \text{on} \quad a<x<b$$

Can we ~~We can~~ convert this 2nd order homogeneous linear D.E into a Sturm-Liouville equation?

$$\begin{cases} (p(x)\phi')' + q(x)\phi + \lambda\, r(x)\,\phi = 0 \\ p', p, q, r \quad \text{continuous on } a \le x \le b \\ p, r > 0 \quad \text{on } a \le x \le b \end{cases}$$

→ We need to find $p(x)$ so that

$$(p(x)\phi')' = p\phi'' + p'\phi' = A_2\phi'' + A_1\phi'$$

$p(x) = A_2(x)$
$p'(x) = A_1(x)$ ⟹ but A_2' may not be A_1.

Look for an integrating factor $\mu(x)$ so

$(\mu A_2)' = \mu A_1$ set $p = \mu A_2$ → $\mu = \frac{p}{A_2}$

$p' = \mu A_1$ → $p' = \frac{p}{A_2} A_1$

→ solve $\int \frac{p'}{p} = \int \frac{A_1}{A_2}$

exp $\left(\ln p = \int \frac{A_1}{A_2}\, dx \right)$ ⟹ $p(x) = c\, e^{\int \frac{A_1}{A_2} dx}$

↳ take $c = 1$

Say something about why $c=1$ is ok

then $\mu(x) = \frac{1}{A_2} e^{\int \frac{A_1}{A_2} dx}$

17-6 check that it all works.

* multiply D.E. by μ

$$\mu A_2 \phi'' + \mu A_1 \phi' + \mu A_0 \phi + \lambda \mu \rho \phi = 0$$

$$(\mu A_2 \phi')' + \mu A_0 \phi + \lambda \mu \rho \phi = 0 \longrightarrow \text{Sturm-Liouville Form}$$

$$(p\phi')' + q\phi + \lambda r \phi = 0$$

$p = \mu A_2$

$q = \mu A_0$

$r = \mu \rho$

we need these to be continuous & $A_2 \neq 0$. } Regularity Condition

---

Ex. $x^2 y'' + x y' + \lambda y$ Cauchy-Euler (i.e. circular disk-)

$y(1) = 0$

$y(b) = 0$ $1 < x < b$.

? Sturm-Liouville Form.

$A_2 = x^2$

$A_1 = x$

$$\mu(x) = \frac{1}{x^2} e^{\int \frac{x}{x^2} dx}$$

$$= \frac{1}{x^2} e^{\int \frac{1}{x} dx}$$

$$= \frac{1}{x^2} e^{\ln x} = \frac{1}{x}$$

$$\frac{1}{x}\left(x^2 y'' + x y' + \lambda y \right) = 0$$

$$x y'' + y' + \frac{\lambda}{x} y = 0$$

$$(x y')' + \frac{\lambda}{x} y = 0$$

weight function is $r(x) = \frac{1}{x}$

$p(x) = x$, $p' = 1$

$q(x) = 1$ are all continuous on

$\sigma(x) = \frac{1}{x}$ $1 \leq x \leq b$

also $p(x)$ & $\sigma(x)$ are > 0.

Ex 1 (fr §11.3, ex 1)

Convert the following eqn into the form of a Sturm-Liouville eqn:

$$3x^2y''(x) + 4xy'(x) + 6y(x) + \lambda y(x) = 0, \quad x>0$$

Ans:

$A_2(x) = 3x^2$, $A_1(x) = 4x$, $A_0(x) = 6$, $\rho(x) = 1$

$>0 \because x>0$

$$\mu(x) = \frac{1}{A_2(x)} e^{\int \frac{A_1(x)}{A_2(x)}dx} = \frac{1}{3x^2} e^{\int \frac{4x}{3x^2}dx} = \frac{1}{3x^2} e^{\frac{4}{3}\int \frac{1}{x}dx}$$

$$= \frac{1}{3x^2} e^{\frac{4}{3}\ln(x) + \cancel{K}^{\,0}} = \frac{1}{3x^2} e^{\ln(x^{4/3})} = \frac{1}{3x^2} x^{4/3} = \frac{1}{3x^{2/3}}$$

Multiplying the ODE by $\mu(x)$ we have

$$\underbrace{\underbrace{x^{4/3}}_{\text{ok},\because x\neq 0} y''(x) + \frac{4}{3}x^{1/3}y'(x)} + \underbrace{2x^{-2/3}}_{\text{ok},\because x\neq 0} y(x) + \lambda \underbrace{\frac{1}{3}x^{-2/3}}_{\text{ok},\because x\neq 0} y(x) = 0$$

$$\left(x^{4/3}y'(x)\right)' + 2x^{-2/3}y(x) + \lambda \frac{1}{3}x^{-2/3}y(x) = 0$$

where we can now identify the S-L coefficients:

$$p(x) = x^{4/3}, \quad q(x) = 2x^{-2/3}, \quad r(x) = \frac{1}{3}x^{-2/3}$$

Ex2 Consider the following BVP

$$y''+6y'+\lambda y=0 \qquad 0<x<1$$

$$y'(0)=0, \quad y(1)=0$$

Convert this equation into a Sturm-Liouville equation and verify that the BVP is regular.

Solution:

$A_2(x)=1$

$A_1(x)=6$

*Compute the integrating factor

$$\mu(x)=\frac{1}{A_2(x)}e^{\int \frac{A_1(x)}{A_2(x)}dx}$$

$$=e^{\int 6dx}$$

$$=e^{6x}$$

*Multiplying $y''+6y'+\lambda y=0$ by $\mu(x)=e^{6x}$ yields

$$e^{6x}y''+6e^{6x}y'+\lambda e^{6x}y=0$$

$$(e^{6x}y')'+\lambda e^{6x}y=0 \Leftarrow \text{Sturm-Liouville Equation}$$

*Recall that for the BVP

$$(p(x)y'(x))'+q(x)y(x)+\lambda r(x)y(x)=0 \qquad a<x<b$$

to be regular, we require that

① $p(x)$, $p'(x)$, $q(x)$, and $r(x)$ are real-valued continuous functions on $[a,b]$

② $p(x)>0$, $r(x)>0$ on $[a,b]$.

*Our BVP is regular since

① $p(x)=e^{6x}$, $p'(x)=6e^{6x}$, $q(x)=0$, $r(x)=e^{6x}$ are real-valued continuous functions on $[0,1]$

② $p(x)=r(x)=e^{6x}>0$ on $[0,1]$

Midterm 2 Cutoff

§11.2 #8 – Transform the given equation into the form $[p(x)y']' + q(x)y = 0$

#8. L'Hermite Equation $\qquad y'' - 2xy' + \lambda y = 0$ $\qquad$ not self-adjoint.

$A_2(x) = 1 \neq 0$, $A_1(x) = -2x$ $\quad$ all coefficients are continuous

$$\mu(x) = \exp\left(\int -2x\,dx\right) = \exp(-x^2)$$

so $\quad e^{-x^2}y'' - 2xe^{-x^2}y' + \lambda e^{-x^2}y = 0 \Rightarrow [e^{-x^2}y']' + \lambda e^{-x^2}y = 0$ $\quad$ this is self adjoint.

$p(x) = e^{-x^2}$

affect the orthogonality condition $r(x) = e^{-x^2}$

#9 – Bessel Equation $\qquad x^2y'' + xy' + (x^2 - \nu^2)y = 0$

$A_2 = x^2 \neq 0 \quad A_1 = x \quad$ all continuous

$$\mu(x) = \frac{1}{x^2}\exp\left(\int \frac{x}{x^2}\,dx\right) = \frac{1}{x^2}e^{\ln x} = \frac{1}{x}$$

so $\quad xy'' + y' + \frac{x^2 - \nu^2}{x}y = 0 \Rightarrow (xy')' + \left(x - \frac{\nu^2}{x}\right)y = 0$

#10 – Laguerre Equation $\qquad xy'' + (1-x)y' + \lambda y = 0$

$$\mu(x) = \frac{1}{x}\exp\left(\int \frac{1-x}{x}\,dx\right) = \frac{1}{x}\exp\left(\int \frac{1}{x} - 1\,dx\right)$$

$$= \frac{1}{x}\exp(\ln x - x) = \frac{1}{x}e^{\ln x}e^{-x} = e^{-x}$$

so $\quad xe^{-x}y'' + (1-x)e^{-x}y' + \lambda e^{-x}y = 0 \Rightarrow [xe^{-x}y']' + \lambda e^{-x}y = 0$

#11 – Chebyshev Equation $\qquad (1-x^2)y'' - xy' + \alpha^2 y = 0$

$$\mu(x) = \frac{1}{1-x^2}\exp\left(\int \frac{-x}{1-x^2}\,dx\right) = \frac{1}{1-x^2}\exp\left(\ln(1-x^2)^{1/2}\right) = \frac{1}{\sqrt{1-x^2}}$$

let $u = 1-x^2$, $du = -2x\,dx$ $\Rightarrow \int \frac{-x}{1-x^2}\,dx = \frac{1}{2}\int \frac{-2x\,dx}{1-x^2} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(1-x^2)$.

so $\quad \sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{\alpha^2}{\sqrt{1-x^2}}y = 0$

$$\Rightarrow [\sqrt{1-x^2}\,y']' + \frac{\alpha^2}{\sqrt{1-x^2}}y = 0$$

$\frac{1}{2\sqrt{1-x^2}}(-2x)y' + \sqrt{1-x^2}\,y''$

§11.3 cont'd

We can think of a DE in terms of a linear differential operator:

$$L[y](x) := (p(x)\,y'(x))' + q(x)\,y(x) \quad \cdots\cdots \quad (1)$$

So our S-L eqn is

$$L[y](x) + \lambda r(x)\,y(x) = 0. \quad \cdots\cdots \quad (2)$$

Thm 1: Lagrange's Identity for $L[y] = (py')' + qy$

Let u & v be fns w/ continuous 2^{nd} derivatives on $[a,b]$. Then

$$uL[v] - vL[u] = \frac{d}{dx}\left(pW[u,v]\right) \quad \cdots \quad (3)$$

where $W[u,v] = uv' - vu'$ is the Wronskian of u + v.

Proof:

$$uL[v] - vL[u] = u[(pv')' + qv] - v[(pu')' + qu]$$
$$= u(pv')' + \cancel{quv} - v(pu')' - \cancel{quv}$$
$$= u(pv')' + \boxed{u'(pv') - v'(pu')} - v(pu')'$$

same, just rearranged

$$= [u(pv')]' - [v(pu')]'$$
$$= [p(uv' - vu')]'$$
$$= \frac{d}{dx}[pW[u,v]]$$

$\blacksquare$

Integrate Lagrange formula to get Green's formula:

$$\int_a^b \left(uL[v] - vL[u]\right)(x)\,dx = \left(pW[u,v]\right)(x)\Big|_a^b \quad \cdots \quad (4)$$

Recall that $W[u,v](x) = u(x)v'(x) - u'(x)v(x)$ so $pW[u,v](x)\Big|_a^b = p\big[u(b)v'(b) - u'(b)v(b) - u(a)v'(a) + u'(a)v(a)\big]$ (see next p)

If u & v satisfy the BCs

$$\begin{cases} a_1 y(a) + a_2 y'(a) = 0 \\ b_1 y(b) + b_2 y'(b) = 0 \end{cases} \quad \cdots \quad (5)$$

then (4) becomes

$$\int_a^b \left(uL[v] - vL[u]\right)(x)\,dx = 0 \quad \cdots \quad (6)$$

We can write (6) in terms of an inner product. Setting

$$\underbrace{(f,g) = \int_a^b f(x)g(x)\,dx}$$

this is an $\boxed{\text{inner product}}$ on the set $f, g \rightarrow$ continuous, real-valued on $[a,b]$. (C^2 fns that satisfy the BCs)

Then (6) becomes

$$\boxed{(u, L[v]) = (L[u], v).} \quad \cdots \quad (7)$$

(pf: see 2nd of following two pages)

In this case we say that $\boxed{L \text{ is self-adjoint.}}$
Self-adjoint operators have special properties (they are like symmetric matrices (real eigenvalues, full set of lin. indep eigenvectors, if eigenvalues are simple then e-vectors form an $\perp$ set.)

Pf of (6):

$$W[u,v](x)\Big|_a^b = u(b)v'(b) - u'(b)v(b) - u(a)v'(a) + u'(a)v(a) \quad \ldots (5a)$$

Now use

$$\begin{cases} a_1 u(a) + a_2 u'(a) = 0 \\ a_1 v(a) + a_2 v'(a) = 0 \end{cases} \iff \begin{cases} u'(a) = -\frac{a_1}{a_2} u(a) \\ v'(a) = -\frac{a_1}{a_2} v(a) \end{cases} \quad \ldots (5b)$$

and

$$\begin{cases} b_1 u(b) + b_2 u'(b) = 0 \\ b_1 v(b) + b_2 v'(b) = 0 \end{cases} \iff \begin{cases} u'(b) = -\frac{b_1}{b_2} u(b) \\ v'(b) = -\frac{b_1}{b_2} v(b) \end{cases} \quad \ldots (5c)$$

Plug (5b) & (5c) into (5a):

$$W(u,v)(x)\Big|_a^b = u(b)\left(-\frac{b_1}{b_2}\right)v(b) + \frac{b_1}{b_2} u(b)v(b) - u(a)\left(-\frac{a_1}{a_2}\right)v(a) + \left(-\frac{a_1}{a_2}\right)u(a)v(a)$$

$$= -\frac{b_1}{b_2} u(b)v(b) + \frac{b_1}{b_2} u(b)v(b) + \frac{a_1}{a_2} u(a)v(a) - \frac{a_1}{a_2} u(a)v(a)$$

$$= 0$$

$\blacksquare$

pf of (7):

$$\int_a^b \left(uL[v] - vL[u]\right)(x)\,dx = 0 \iff \int_a^b uL[v](x)\,dx = \int_a^b vL[u](x)\,dx \iff //$$

$$// \iff (u, L[v]) = (v, L[u])$$

§11.3 Cont'd

Other properties of the eigenvalues & eigenfunctions of a regular S-L BVP:

(i) The eigenvalues & eigenfns are real-valued

(ii) The eigenvalues are simple

(iii) Sequence of eigenvalues: $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ w/ $\lim_{n\to\infty} \lambda_n = \infty$

Pf of (ii)

Assume $\phi(x)$ & $\psi(x)$ are two eigenfns corresponding to the eigenvalue λ. So $\phi(x)$ & $\psi(x)$ are solutions to the same ODE. They satisfy the BCs:

$$\begin{cases} a_1 y(a) + a_2 y'(a) = 0, \\ b_1 y(b) + b_2 y'(b) = 0, \end{cases}$$

so we have

$$a_1 \phi(a) + a_2 \phi'(a) = 0, \quad a_1 \psi(a) + a_2 \psi'(a) = 0.$$

Let's assume that $a_2 \neq 0$. Solving for $\phi'(a)$ & $\psi'(a)$ gives

$$\phi'(a) = -\frac{a_1}{a_2}\phi(a), \quad \psi'(a) = -\frac{a_1}{a_2}\psi(a).$$

Now compute the Wronskian of ϕ & ψ at $x=a$:

$$W[\phi,\psi](a) = \phi(a)\psi'(a) - \phi'(a)\psi(a)$$
$$= \phi(a)\left(-\frac{a_1}{a_2}\right)\psi(a) + \frac{a_1}{a_2}\phi(a)\psi(a) = 0$$

Thus, $\phi(x)$ & $\psi(x)$ are linearly dependent, so one must be a multiple of the other, & λ is a simple eigenvalue.

§11.4 Non-homogeneous BVPs

Homogeneous BVPs $\rightarrow$ always have at least $y=0$ as a sol'n

Nonhom. BVPs $\rightarrow$ may not have any sol'n!

We need a test

The general non-hom problem:

$$A_2(x)\,y''(x) + A_1(x)\,y'(x) + A_0(x)\,y(x) = h(x), \quad a<x<b \quad \ldots (1)$$

$$a_{11}y(a) + a_{12}y'(a) + b_{11}y(b) + b_{12}y'(b) = 0 \quad \ldots (2)$$

$$a_{21}y(a) + a_{22}y'(a) + b_{21}y(b) + b_{21}y'(b) = 0 \quad \ldots (3)$$

When does this BVP have a sol'n?

Defn Formal Adjoint

Let L be the differential operator

$$L[y] := A_2 y'' + A_1 y' + A_0 y. \quad \ldots (4)$$

The formal adjoint (or Lagrange adjoint) of L is the differential operator L^+ defined by

$$L^+[y] := (A_2 y)'' - (A_1 y)' + A_0 y \quad \ldots (5)$$

provided that A_2'', A_1', and A_0 are continuous on $[a,b]$.

Note: If the ODE is in S-L form then $L = L^+$.

Ex 1

Determine the formal adjoint of
$$L[y] = y'' + 2y' + 2y.$$

Sol'n

$A_2 = 1,\ A_1 = 2,\ \&\ A_0 = 2$

$$L^+[y] = (A_2 y)'' - (A_1 y)' + A_0 y$$
$$= (1y)'' - (2y)' + 2y = y'' - 2y' + 2y$$

(note: $L \neq L^+$ $\therefore$ the ODE is not in S-L form)

Ex 2

Determine the formal adjoint of
$$L[y](x) = x^2 y''(x) + xy'(x) + 4y(x)$$

Sol'n

$A_2(x) = x^2,\ A_1(x) = x,\ A_0(x) = 4$

$$\therefore L^+[y](x) = (x^2 y(x))'' - (xy(x))' + 4y(x)$$
$$= (2xy(x) + x^2 y'(x))' - (y(x) + xy'(x)) + 4y(x)$$
$$= x^2 y''(x) + 3xy'(x) + 3y(x)$$

L & L^+ (the formal adjoint) are related by a generalisation of the Lagrange Identity for S-L problems:

Thm 9: Lagrange's Identity

Let L be the differential operator
$$L[y] = A_2 y'' + A_1 y' + A_0 y$$
and let L^+ be its formal adjoint. Then
$$L[u]v - uL^+[v] = \frac{d}{dx}\left[P(u,v)\right], \quad \cdots\cdots (6)$$
where $P(u,v)$ is the bilinear concomitant associated with L, is defined as
$$P(u,v) := uA_1v - u(A_2v)' + u'A_2v.$$

Pf

Consider (6).

$$\begin{aligned} LHS &= L[u]v - uL^+[v] \\ &= (A_2u'' + A_1u' + A_0u)v - u\left((A_2v)'' - (A_1v)' + A_0v\right) \\ &= A_2u''v + A_1u'v + \cancel{A_0uv} - u(A_2v)'' + u(A_1v)' - \cancel{A_0uv} \\ &= A_2u''v + A_1u'v + \quad - u(A_2v)'' + u(A_1v)' \end{aligned}$$

$$\begin{aligned} RHS &= \frac{d}{dx}\left[uA_1v - u(A_2v)' + u'A_2v\right] \\ &= u'A_1v + u(A_1v)' - \cancel{u'(A_2v)'} \\ &\quad - u(A_2v)'' + u''A_2v + \cancel{u'(A_2v)'} \end{aligned}$$

$\therefore$ LHS = RHS

As before, integrating Lagrange's Identity leads to Green's Formula:

Corollary 2: Green's Formula

Let L be the differential operator (4), and let L^+ be its formal adjoint. Then

$$\int_a^b (L[u]v - uL^+[v])(x)\,dx = P(u,v)(x)\Big|_a^b \quad \cdots (7)$$

or

$$(L[u],v) = (u, L^+[v]) + P(u,v)\Big|_a^b \quad \cdots (8)$$

Note that L^+ & P can be derived from $(L[u],v)$:

$$(L[u],v) = \int_a^b L[u](x)\,v(x)\,dx$$

$$= \int_a^b (A_2u'' + A_1u' + A_0u)v\,dx$$

(IBP) $\vdots$

$$= \int_a^b u\underbrace{\left((A_2v)'' - (A_1v)' + A_0v\right)}_{L^+[v] \text{ (formal adjoint)}}dx$$

$$+ \underbrace{\left[uA_1v + u'A_2v - u(A_2v)'\right]}_{\substack{P(u,v) \\ \text{(bilinear concomitant)}}}$$

Recall that, in §11.3, we defined an inner product

$$(f,g) = \int_a^b f(x)\, g(x)\, dx . \qquad \cdots \quad (9)$$

So we can rewrite Green's formula:

$$(L[u],v) = (u, L^+[v]) + P(u,v)\Big|_a^b \qquad \cdots \quad (10)$$

In applications we need a few more restricOs. For example, in solving BVPs, we apply L only to those fns that satisfy the given BCs. So, we need to consider the domains, $D(L)$ and $D(L^+)$ of the operators L and L^+.

Useful choice: Restrict L^+ to a domain $D(L^+)$ so that

$$(L[u],v) = (u, L^+[v]), \qquad \cdots \quad (11)$$

adjoint operator (drop "formal")

so $P(u,v)\Big|_a^b = 0$ for u in $D(L)$ and v in $D(L^+)$ $\quad \cdots \quad$ (12)

Special Case:

If $L^+ = L$, and (11) with $D(L^+) = D(L)$.

then L is <u>self-adjoint</u>.

Def: Adjoint BVP

Let $B[u]=0$ be the BCs required for a function to be in the domain of L, and let $B^+[v]=0$ be the BCs required for a function to be in the domain of L^+. Then the adjoint BVP for

$$L[u]=0, \quad B[u]=0$$

is the BVP

$$L^+[v]=0, \quad B^+[v]=0.$$

Ex 3 Find the adjoint BVP for

$$y''-2y'+10y=0$$
$$y(0)=y(\pi)=0$$

Solution:

We are given

$L[y]$: $y''-2y'+10y=0$

$B[y]$: $y(0)=y(\pi)=0$.

We have

$A_2=1, \quad A_1=-2, \quad A_0=10$

So the adjoint operator for L is

$$\begin{aligned} L^+[y] &= (A_2 y)''-(A_1 y)'+A_0 y \\ &= (1\cdot y)''-(-2y)'+10y \\ &= y''+2y'+10y=0 \end{aligned}$$

To find the adjoint BCs, we need

$$P(u,v)(x)\Big|_0^{\pi}=0 \qquad \text{where } u\in D(L), \; v\in D(L^+),$$

and

$$P(u,v) = uA_1v-u(A_2v)'+u'A_2v.$$

$$P(u,v)\Big|_0^{\pi} = \big(2u(\pi)v(\pi) - u(\pi)v'(\pi) + u'(\pi)v(\pi)\big) - \big(-2u(0)v(0) - u(0)v'(0) + u'(0)v(0)\big) = 0$$

Since $u \in D(L)$, we have

$$u(0) = 0, \quad u(\pi) = 0.$$

Day 19 starts here.

$$\therefore P(u,v)\Big|_0^{\pi} = u'(\pi)v(\pi) - u'(0)v(0) = 0$$

Because $u'(\pi)$ & $u'(0)$ are arbitrary, we must have

$$v(\pi) = v(0) = 0.$$

Thus, the adjoint BVP is

$$L^+[y]: \; y'' + 2y' + 10y = 0$$
$$B^+[y]: \; y(0) = y(\pi) = 0$$

save this result (+ orig problem) on another board

Ex 4 Find the adjant BVP for

$$x^2y''(x) + 3xy'(x) + 2y(x) = 0$$
$$y'(1) = y'(e^{\pi}) = 0$$

Solution:

We are given

$$L[y]: \quad x^2y''(x) + 3xy'(x) + 2y(x) = 0$$
$$B[y]: \quad y'(1) = y'(e^{\pi}) = 0.$$

We have

$$A_2 = x^2, \quad A_1 = 3x, \quad A_0 = 2$$

So the adjoint operater for L is

$$L^+[y] = (A_2y)'' - (A_1y)' + A_0y$$
$$= (x^2y)'' - (3xy)' + 2y$$
$$= (2xy + x^2y')' - (3y + 3xy') + 2y$$

$$L^+[y] = 2y + 2xy' + 2xy' + x^2y'' - 3y - 3xy' + 2y$$

$$= x^2y'' + xy' + y = 0$$

To find the adjoint B'Cs, we need

$$P(u,v)(x)\Big|_1^{e^\pi} = 0 \qquad \text{where} \quad u \in D(L),\ v \in D(L^+)$$

and

$$P(u,v) = uA_1v - u(A_2v)' + u'A_2v$$

$$= u(3x)v - u(x^2v)' + u'x^2v$$

$$= 3xuv - u(2xv + x^2v') + x^2u'v$$

$$= xuv - x^2(uv' - u'v)$$

$$P(u,v)(x)\Big|_1^{e^\pi} = 0$$

$$\left[e^\pi u(e^\pi)v(e^\pi) - e^{2\pi}\left(u(e^\pi)v'(e^\pi) - \cancelto{0}{u'(e^\pi)}v(e^\pi)\right)\right] -$$

$$\left[1 \cdot u(1)v(1) - 1^2\left(u(1)v'(1) - \cancelto{0}{u'(1)}v(1)\right)\right] = 0$$

Since $u \in D(L)$, we have

$$u'(1) = u'(e^\pi) = 0$$

$$\therefore\ e^\pi u(e^\pi)v(e^\pi) - e^{2\pi}u(e^\pi)v'(e^\pi) - u(1)v(1) - u(1)v'(1) = 0$$

$$e^\pi u(e^\pi)\left[v(e^\pi) - e^\pi v'(e^\pi)\right] - u(1)\left[v(1) - v'(1)\right] = 0$$

Because $u(1)$ & $u(e^\pi)$ are arbitrary, we must have

$$v(e^\pi) - e^\pi v'(e^\pi) = 0$$

and $$v(1) - v'(1) = 0$$

Thus, the adjoint BVP is

$L^+[y]:\ x^2y'' + xy' + y = 0$

$B^+[y]:\ y(e^\pi) - e^\pi y'(e^\pi) = 0$

$y(1) - y'(1) = 0.$

Save this result (& orig prob'm) on a separate board

* We want to determine when the nonhomogeneous BVP

$$L[u]=h$$
$$B[u]=0$$

has a solution.

Assume v is a solution to the homogeneous adjoint problem.

$$\left.\begin{array}{l} L^+[v]=0 \\ B^+[v]=0 \end{array}\right\} \cdots\cdots (*)$$

Consider

$$(h,v)=\int_a^b h(x)v(x)\,dx.$$

Notice that

$$\begin{aligned}(h,v)&=(L[u],v) && \text{since } L[u]=h\\ &=(u,L^+[v]) && \text{by definition of the formal adjoint}\\ &=(u,0) && \text{since } L^+[V]=0.\\ &=0,\end{aligned}$$

so we need

$$(h,v)=0 \qquad \forall\ v \text{ satisfying } (*).$$

In other words, for the nonhomogeneous problem to have a solution, the nonhomogeneous term h must be orthogonal to <u>all</u> solutions of the adjoint problem.

* This test is called the <u>Fredholm Alternative</u> *

Theorem: Fredholm Alternative

Let L be a linear differential operator and let B represent a set of linear BCs. The nonhomogeneous BVP

$$L[y](x) = h(x), \quad a<x<b,$$
$$B[y] = 0$$

has a solution iff

$$\int_a^b h(x)\, z(x)\, dx = 0$$

for every solution z of the adjoint BVP

$$L^+[z](x) = 0, \quad a<x<b,$$
$$B^+[z] = 0.$$

State Remark on Page 95 next

Ex 3 (continued).
Determine conditions on h that guarantee that the given nonhomogeneous BVP has a solution

$$y'' - 2y' + 10y = h(x)$$
$$y(0) = y(\pi) = 0.$$

Solution:

We already deduced that the adjoint BVP for the associated homogeneous BVP is

$$L^+[y]: y'' + 2y' + 10y = 0$$
$$B^+[y]: y(\pi) = y(0) = 0.$$

To apply the theorem, we need to solve the homogeneous problem $L^+[y] = 0, \; B^+[y] = 0.$

94)

Let $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

Then

$y'' + 2y' + 10y = 0$

$r^2 e^{rx} + 2re^{rx} + 10e^{rx} = 0$

$(r^2 + 2r + 10)e^{rx} = 0$

$r^2 + 2r + 10 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

The general solution is

$y(x) = c_1 e^{-x} \cos(3x) + c_2 e^{-x} \sin(3x)$

$y(0) = c_1 = 0 \quad \rightarrow \quad y(x) = c_2 e^{-x} \sin(3x)$

$y(\pi) = c_2 e^{-\pi} \sin(3\pi) = 0$ ✓ Trivially Satisfied.

$\therefore$ Every solution of the adjoint problem has the form

$y(x) = c_2 e^{-x} \sin(3x)$, where c_2 is free

It follows from the Fredholm alternative that if h is continuous, then the nonhomogeneous problem has a Solution iff

$$\int_0^{\pi} h(x) e^{-x} \sin(3x)\, dx = 0.$$

EX ~~4~~ (continued)

Determine conditions on h that guarantee that the nonhomogeneous BVP has a solution.

$$x^2y''(x)+3xy'(x)+2y(x)=h(x)$$

$$y'(1)=y'(e^{\pi})=0.$$

Solution:

We already deduced that the adjoint BVP for the associated homogeneous BVP is

$$L^+[y]: x^2y''+xy'+y=0$$

$$B^+[y]: y(e^{\pi})-e^{\pi}y'(e^{\pi})=0$$

$$y(1)-y'(1)=0.$$

To apply the Fredholm alternative, we need to solve the homogeneous problem

$$L^+[y]=0, \quad B^+[y]=0$$

Since $x^2y''+xy'+y=0$ is a Cauchy-Euler equation, Set

$$y=x^r$$

$$y'=rx^{r-1}$$

$$y''=r(r-1)x^{r-2}$$

Then

$$x^2r(r-1)x^{r-2}+xrx^{r-1}+x^r=0$$

$$(r^2-r+r+1)x^r=0$$

$$r^2+1=0 \qquad \left(\text{Since } x^r=0 \text{ only gives the trivial solution } y(x)=0\right)$$

$$r^2=-1$$

$$r=\pm i$$

The general solution is

$$y(x)=C_1\cos(\ln x)+C_2\sin(\ln x).$$

95) $y'(x) = -c_1 \frac{1}{x}\sin(\ln x) + c_2 \frac{1}{x}\cos(\ln x)$

$$y(e^\pi) - e^\pi y'(e^\pi) = 0$$

$$c_1\cos(\ln e^\pi) + c_2 \cancel{\sin(\ln e^\pi)}^{0} - e^\pi\left(-\frac{c_1}{e^\pi}\cancel{\sin(\ln e^\pi)}^{0} + \frac{c_2}{e^\pi}\cos(\ln e^\pi)\right) = 0$$

$$c_1\cos(\pi) - c_2\cos(\pi) = 0$$

$$(c_1 - c_2)\cos(\pi) = 0$$

$$c_1 = c_2.$$

$$y(1) - y'(1) = 0$$

$$c_1\cos(\ln(1)) + c_1\cancel{\sin(\ln(1))}^{0} - \left(-\frac{c_1}{1}\cancel{\sin(\ln(1))}^{0} + c_1\frac{1}{1}\cos(\ln(1))\right) = 0$$

$c_1\cos(0) - c_1\cos(0) = 0$ ✓ Trivially Satisfied

∴ Every solution of the adjoint problem has the form

$$y(x) = c_1(\cos(\ln x) + \sin(\ln x))$$

where c_1 is free.

It follows from the Fredholm alternative that if h is continuous, then the nonhomogeneous problem has a solution iff

$$\int_1^{e^\pi} h(x)(\cos(\ln x) + \sin(\ln x))\,dx = 0.$$

Remark: State this remark after Fredholm.

If the adjoint BVP stated in the Fredholm alternative theorem has only the trivial solution, $z = 0$, then the nonhomogeneous BVP has a solution for each h

Since $\int_a^b h(x)z(x)\,dx = 0 \quad \forall\, h(x)$ when $z(x) = 0$.

If the BCs for the nonhomogeneous problem are either separated or periodic, then it [illegible]n that there is a unique solution for [illegible]

Solution by Eigenfunction Expansion: (Section 11.3)

* The Fredholm alternative provides a test to determine whether solutions exist for a given nonhomogeneous BVP

$\Rightarrow$ How do we find such a solution?

* Recall the eigenfunctions associated with a regular Sturm-Liouville BVP form an orthogonal system
* Our goal is to find an eigenfunction expansion for a solution to the nonhomogeneous regular Sturm-Liouville BVP

$$(*)\begin{cases} L[y]+\mu r y = f \\ a_1 y(a)+a_2 y'(a)=0 \\ b_1 y(b)+b_2 y'(b)=0 \end{cases} \qquad \mu = \text{fixed real number}$$

and

$$L[y]=(py')'+qy.$$

Suppose that $\{\lambda_n\}_{n=1}^{\infty}$ are eigenvalues with corresponding eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ for the homogeneous eigenvalue problem associated with $(*)$, with μ replaced by λ. That is

$$L[\phi_n]+\lambda_n r \phi_n = 0$$

with ϕ_n satisfying the BCs in $(*)$.

We know that $\{\phi_n\}_{n=1}^{\infty}$ forms an orthogonal system with respect to the weight function $r(x)$ on $[a,b]$

Chap 11 - Summary

When we apply separation of variables to a PDE, we obtain at least one Boundary Value Problem. The BVPs that arise from the heat, wave, + Laplace eqns are special cases of Sturm-Liouville BVPs.

S-L BVPs

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0, \quad a < x < b$$

$$a_1 y(a) + a_2 y'(a) = 0, \quad b_1 y(b) + b_2 y'(b) = 0$$

What do we know about Regular S-L BVPs?

1. Any 2^{nd} order ODE can be converted into S-L form
2. The eigenvalues are real, simple, and have real-valued eigenfunctions. They form a countable, increasing sequence.
3. The eigenfunctions are orthogonal with respect to the weight function $r(x)$ on $[a,b]$.

Key Formulas

Let $L[y](x) := (p(x)y'(x))' + q(x)y(x)$

$B[y] = 0$ (BCs for any fn in the domain of L)

Then the S-L eqn is $L[y](x) + \lambda r(x) y(x) = 0$. We have

$$\int_a^b \big(uL[v] - vL[u]\big)(x)\,dx = \rho W[u,v](x)\Big|_a^b .$$

Wronskian

If $B[u]=0$ and $B[v]=0$, then

$$\int_a^b \big(uL[v] - vL[u]\big)(x)\,dx = 0 \iff (u, L[v]) = (L[u], v)$$

and L is self-adjoint.

Nonhomogeneous, Regular S-L BVPs

$$L[y](x) := A_2(x)y''(x) + A_1(x)y'(x) + A_0(x)y(x) = h(x) \qquad a<x<b$$

$$B[y]=0 \begin{cases} B_1[y] := a_{11}y(a) + a_{12}y'(a) + b_{11}y(b) + b_{12}y'(b) = 0 \\ B_2[y] := a_{21}y(a) + a_{22}y'(a) + b_{21}y(b) + b_{22}y'(b) = 0 \end{cases}$$

This problem has a solution iff

$$\int_a^b h(x) z(x)\,dx = 0$$

for every solution z of the adjoint BVP

$$\begin{cases} L^+[z](x) = 0 & a<x<b \\ B^+[z] = 0. \end{cases}$$

where

$$L^+[y] := (A_2 y)'' - (A_1 y)' + A_0 y$$

and $B^+[y]$ is found by solving

$$P(u,v)\Big|_a^b = 0 \quad \text{for } B[u]=0 \text{ and } B^+[v]=0$$

and

$$P(u,v) := u A_1 v - u(A_2 v)' + u' A_2 v$$