

More about ODE - To solve PDEs, we will often (almost
always!) transform the PDE into (no P)
ODEs, & solve those. So let's review... (2)

Simple Examples

1st order linear homogeneous ODE

Ex1 $y' - y = 0 \quad (y' + e^y = 0 \text{ is NOT linear})$

SOL'N: look for y s.t. $y' = y$

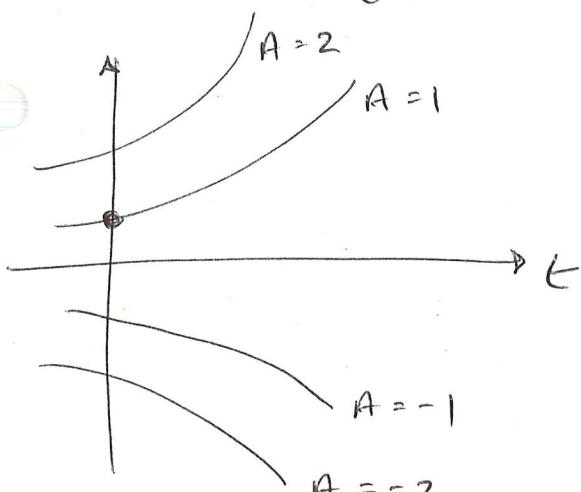
(1) guess: $y(t) = et, y'(t) = et$
 $y(t) = At, y'(t) = Aet$

(2) Separation of Variables:

$$\frac{dy}{dt} - y = 0 \Leftrightarrow \frac{dy}{dt} = y \Leftrightarrow \left\{ \begin{array}{l} \frac{dy}{y} = dt \\ \int \end{array} \right.$$

$$\Leftrightarrow \ln(y) = t + C$$

$$\Leftrightarrow y = e^{t+C} = Ae^t$$



↑ family of solutions (general solution)

Each solution is determined by imposing an initial condition $y(0) = \underline{\hspace{2cm}}$.

2nd order linear homogeneous ODE

Ex2 $y'' - y' - 2y = 0$

Try $y = e^{rt}$, then we obtain
 $y' = re^{rt}$
 $y'' = r^2 e^{rt}$

Plugging into the ODE:

$$r^2 e^{rt} - r e^{rt} - 2 e^{rt} = 0 \Leftrightarrow r^2 - r - 2 = 0$$

$$\Leftrightarrow (r-2)(r+1) = 0$$

$$\Leftrightarrow r=2 \text{ or } r=-1$$

∴ we have two solutions $y_1 = e^{2t}$, $y_2 = e^{-t}$,

and the general solution is

$$y = c_1 e^{2t} + c_2 e^{-t}$$

two-parameter family
of solutions

Each solution is determined by imposing TWO initial conditions:

$$y(t_0) = \underline{\quad}, \quad y'(t_0) = \underline{\quad}.$$

Ex 3] Another important ODE (2nd order, linear)

$$y'' + y = 0 \Leftrightarrow r^2 + 1 = 0 \Leftrightarrow r^2 = -1 \Leftrightarrow r = \pm i$$

$$\therefore y_1(t) = e^{it}, \quad y_2(t) = e^{-it}$$

Euler's formula: $e^{it} = \cos(t) + i\sin(t)$

We find that the real & imaginary parts form a fundamental solution set for the ODE, and so

$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

PDE - A simple Example

Ex. 1. $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$

$$u = u(x, t)$$

1st-order PDE
linear, constant coefficient
Homogeneous.

* guess

Consider $u(x, t) = F(x-t)$

$$\frac{\partial u}{\partial t} = -F'(x-t)$$

$$\frac{\partial u}{\partial x} = F'(x-t)$$

Subst. back in P.D.E

$$-F(x-t) + F(x-t) = 0 \quad \checkmark$$

* solution is any arbitrary function of $x-t$

i.e. $(x-t)^2$, $\sin(x-t)$, e^{x-t} , etc ..

* To choose a particular solution, we need to specify u along some curve in the x, t -plane.

$$u(x, 0) = F(x)$$

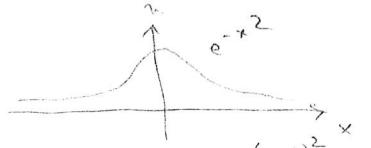
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Say $u(x, 0) = e^{-x^2} : f(x) \Rightarrow u(x, t) = F(x-t)$

$$= e^{-(x-t)^2}$$

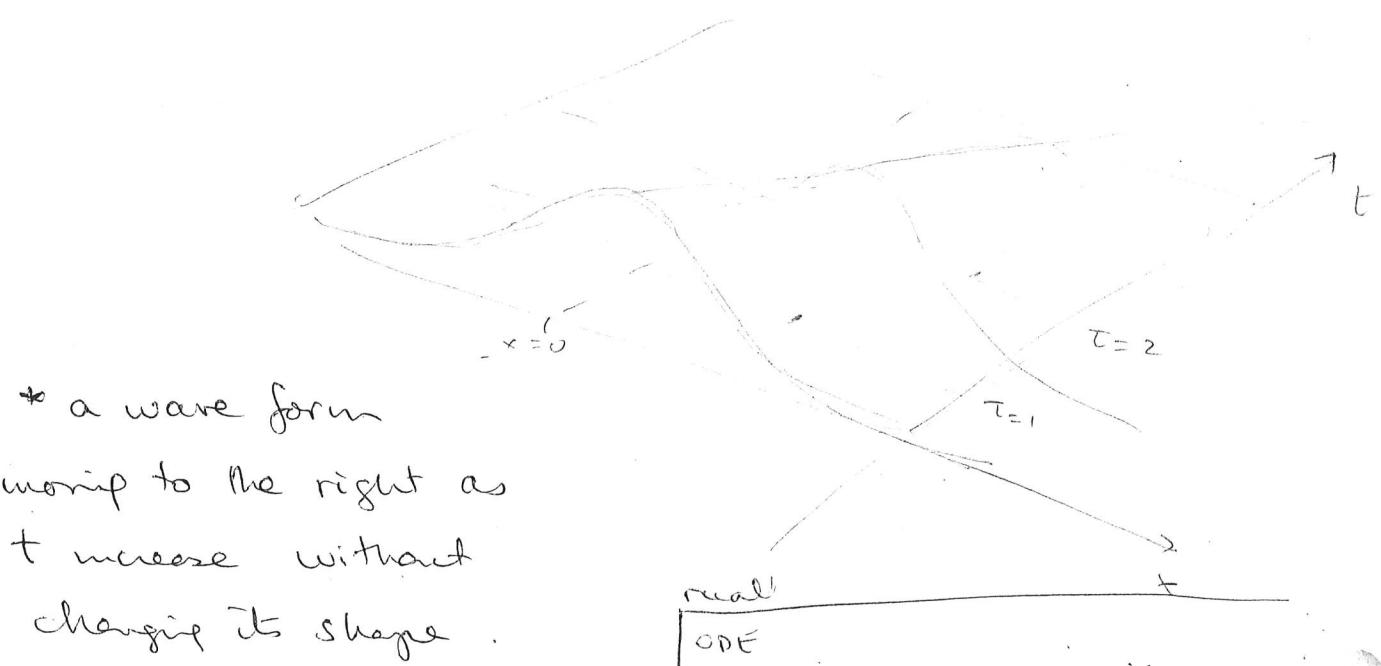
solution.

look at section.

at $t=0$  $t=1$  $t=2$  $t=3$ 

* Could have
 $w(x, t) = A e^{-(x-t)^2}$

solution
is a surface



* a wave form

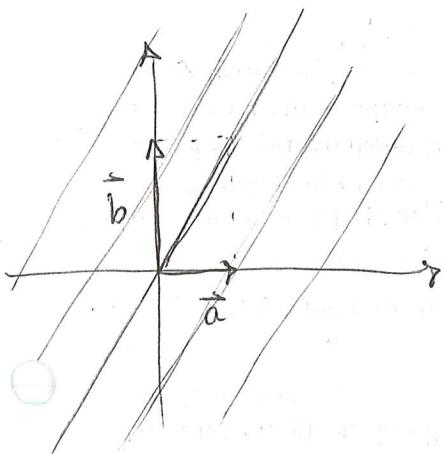
moving to the right as
 t increase without
changing its shape.

real	t
ODE	one degree of freedom \Rightarrow specify one par a-dim & const
PDE	one degree of freedom \Rightarrow specify one curv 1-dim etc

Consider the general constant-coefficient case:

$$\boxed{a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = f(x, y)} \quad \left\{ \begin{array}{l} \text{non-hom,} \\ \text{linear} \\ \text{const coeff} \end{array} \right.$$

$$\left. \begin{aligned} a u_x + b u_y &= (a, b) \cdot (u_x, u_y) \\ &= (\hat{a}^i + \hat{b}^j) \cdot \vec{\nabla} u \end{aligned} \right\}$$



This is the directional derivative of u in the direction of the vector (a, b) .

So we consider a family of lines of slope b/a :

$bx - ay = d$: lines \parallel to $\hat{a}^i + \hat{b}^j$

These are the characteristic lines of the PDE.

Choose new coordinates,

$$\left\{ \begin{array}{l} w = bx - ay, \\ z = y, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x = \frac{1}{b}(w + az), \\ y = z, \end{array} \right.$$

and set $v(w, z) = u(x, y)$. Then

$$\begin{aligned} a u_x + b u_y &= a(v_w w_x + v_z z_x) + b(v_w w_y + v_z z_y) \\ &= ab v_{ww} + b((-a)v_w + v_z) \\ &= b v_{zz} \end{aligned}$$

and the PDE simplifies to

$$b v_{zz} + c v = f\left(\frac{1}{b}(w + az), z\right).$$

Ex (if time) (nextp)

Ex (if time) is used when the "if time" condition is satisfied and the body of the loop is executed. The program continues to execute until the condition becomes false. If the condition is always true, the loop will never end.

The following example illustrates the use of the Ex (if time) command:

```

for i = 1 to 5 do
    print(i)
    if i > 3 then
        break
    else
        print(i * i)

```

In this example, the loop iterates five times. The first iteration prints 1 and then 1. The second iteration prints 2 and then 4. The third iteration prints 3 and then 9. The fourth iteration prints 4 and then 16. The fifth iteration prints 5 and then 25. When the value of i reaches 4, the condition i > 3 becomes true, so the loop is terminated.

Ex (if time) can also be used in a while loop:

```

while i < 5 do
    print(i)
    if i > 3 then
        break
    else
        print(i * i)

```

In this example, the loop iterates five times. The first iteration prints 1 and then 1. The second iteration prints 2 and then 4. The third iteration prints 3 and then 9. The fourth iteration prints 4 and then 16. The fifth iteration prints 5 and then 25. When the value of i reaches 4, the condition i < 5 becomes false, so the loop is terminated.

Ex (if time) is often used in a for loop:

The following example illustrates the use of the Ex (if time) command in a for loop:

```

for i in range(1, 6) do
    print(i)
    if i > 3 then
        break
    else
        print(i * i)

```

In this example, the loop iterates five times.

The following example illustrates the use of the Ex (if time) command in a while loop:

Math 319
Partial Differential Equations
Pre-Lecture Assignment #2
due Thu September 8th, 10am
Tu 13

Instructions: Your work will be marked before class and the solution will be discussed as a group during the first few minutes of the lecture.

1. **Review of ODEs:** Solve the IVP

$$y'' + 2y' + 3y = 0, \quad y(0) = 0, y'(0) = 1.$$

2. **Method of Characteristics:** Consider the PDE

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + 5u = \sin(xy). \quad (1)$$

- (a) Find the preferred direction and family of characteristic lines all parallel to the preferred direction vector.
- (b) Write the equations for the new variables w and z in terms of x and y .
- (c) Rewrite the PDE (1) in terms of the new variables.
- (d) Form the integrating factor

$$\mu(z) = \exp\left(\int \frac{c}{b} dz\right).$$

3. Do you have any questions about the syllabus or other aspect of the course?

Pre-Reading #2 - Sol'n's

1. $y'' + 2y' + 3y = 0$

$$y(0) = 0, \quad y'(0) = 1.$$

char eqn: $r^2 + 2r + 3 = 0 \Leftrightarrow r = -1 \pm \sqrt{1-3} = -1 \pm \sqrt{-2} = -1 \pm i\sqrt{2}$

$$\therefore y(t) = e^{-t} (c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t))$$

apply I.C.s:

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ \left[-e^{-t} \sin(\sqrt{2}t) \right] + e^{-t} \sqrt{2} \cos(\sqrt{2}t) \Big|_{t=0} \end{cases} \Rightarrow c_2 = 1$$

$$\Leftrightarrow \begin{cases} c_1 = 0 \\ \sqrt{2} c_2 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 1/\sqrt{2} \end{cases}$$

$$\therefore y(t) = \frac{e^{-t}}{\sqrt{2}} \sin(\sqrt{2}t)$$

damped oscillations
(underdamped)

2. $3ux_x + 2uy_y + 5u = \sin(\omega y)$

a) preferred direction $w = \frac{2}{3}$ or $\vec{w} = (3\hat{i} + 2\hat{j}) = (3, 2)$

b) $\begin{cases} w = ux_x - uy_y \\ z = uy \end{cases} \Leftrightarrow \begin{cases} w = 2ux - 3uy \\ z = uy \end{cases} \Leftrightarrow \begin{cases} u = \frac{1}{2}(w+3z) \\ y = z \end{cases}$

c) let $v(w, z) = uh(y)$. Then

$$3ux_x + 2uy_y = 2v_z \Rightarrow 2v_z + 5v = \sin\left(\frac{1}{2}(w+3z)\cdot z\right)$$

d) $\mu(z) = e^{\int \frac{5}{2} dz} = e^{\frac{5}{2}z}$ (ignore the cst.)

(2)

(9)

For lecture: Finish the solution
(for part 2)

$$2v_z + 5v = \sin\left(\frac{(w+3z) \cdot z}{2}\right) \Leftrightarrow 1)$$

$$1) \Leftrightarrow \frac{dv}{dz} + \frac{5}{2}v = \frac{1}{2} \sin\left(\frac{(w+3z) \cdot z}{2}\right)$$

$$\Leftrightarrow e^{\frac{5}{2}z} \frac{dv}{dz} + e^{\frac{5}{2}z} v = \frac{1}{2} e^{\frac{5}{2}z} \sin\left(\frac{(w+3z) \cdot z}{2}\right)$$

$$\Leftrightarrow \frac{d}{dz} \left(e^{\frac{5}{2}z} v \right) = \frac{1}{2} e^{\frac{5}{2}z} \sin\left(\frac{(w+3z) \cdot z}{2}\right)$$

$$\Leftrightarrow e^{\frac{5}{2}z} v = \int \frac{1}{2} e^{\frac{5}{2}z} \sin\left(\frac{(w+3z) \cdot z}{2}\right) dz + C(w)$$

$$\Leftrightarrow v = e^{-\frac{5}{2}z} \int \frac{1}{2} e^{\frac{5}{2}z} \sin\left(\frac{(w+3z) \cdot z}{2}\right) dz + e^{-\frac{5}{2}z} C(w)$$

To determine $C(w)$ need λ

(3)

(10)

In terms of x and y we have

$$u(x,y) = e^{-\frac{5}{2}y} \int_{\frac{1}{2}}^{\frac{5}{2}y} e^{\frac{5}{2}y} \sin(-xy) dy$$

$$+ C(2x - 3y)$$

$$= \frac{e^{-\frac{5}{2}y}}{2} \int e^{\frac{5}{2}y} \sin(xy) dy + C(2x - 3y)$$

In order to determine C , we require $u(x_0)$ or $u(0, y)$.