Sample Final Exam Solutions

/60

12 pts

1) a) Find a formal solution to the following vibrating string problem.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{in } 0<x<1 \text{ and } t>0$$

$$u(0,t)=u(1,t)=0 \quad \text{for } t>0,$$

$$u(x,0)=2\sin(\pi x) \quad \text{for } 0<x<1,$$

$$u_t(x,0)=0 \quad \text{for } 0<x<1.$$

When you go through and check the cases, cases 1 & 2 give the trivial solution, so do not check them. Only check the case where $\lambda>0$.

b) Give physical interpretations for the BCs at $x=0$ & $x=1$, and the ICs at $t=0$.

Solution:

a) *Apply the Method of Separation of Variables

Step 1:

Assume $u(x,t)=F(x)G(t)$

Substitution into the PDE yields (1)

$$F(x)G''(t)=F''(x)G(t)$$

$$\underbrace{\frac{G''(t)}{G(t)}}_{\text{constant wrt } x} = \underbrace{\frac{F''(x)}{F(x)}}_{\text{constant wrt } t} = -\lambda \qquad \lambda = \text{constant}$$

Step 2: Set up the 2 ODEs

① $\frac{F''(x)}{F(x)} = -\lambda \iff F''(x)+\lambda F(x)=0, \quad F(0)=F(1)=0$ (2)

② $\frac{G''(t)}{G(t)} = -\lambda \iff G''(t)+\lambda G(t)=0, \quad G'(0)=0$

We have homogeneous BCs in x:

$u(0,t) = F(0)G(t) = 0$
$u(1,t) = F(1)G(t) = 0$ $\quad t>0 \implies F(0) = F(1) = 0$

We also have one homogeneous IC.

$u_t(x,0) = F(x)G'(0) = 0 \quad 0<x<1 \implies G'(0) = 0$

Step 3: Solve the 2 ODEs

* Solve ① first to find the eigenvalues.

Let
$F(x) = e^{rx}$
$F'(x) = re^{rx}$
$F''(x) = r^2e^{rx}$

Then
$r^2e^{rx} + \lambda e^{rx} = 0$
$(r^2+\lambda)e^{rx} = 0$
$r^2 = -\lambda$

* We are told that cases 1 & 2 yield the trivial solution, so we only need to check case 3.

Case 3: $\lambda > 0$, so set $\lambda = \mu^2$ with $\mu > 0$.
Then $r^2 = -\mu^2 \rightarrow r = \pm i\mu$

$F(x) = C_3\cos(\mu x) + C_4\sin(\mu x)$

$F(0) = C_3 = 0 \rightarrow F(x) = C_4\sin(\mu x)$ ②

$F(1) = C_4\sin(\mu) = 0$
$\sin(\mu) = 0$
$\mu = n\pi$ for $n = 1,2,3,\ldots$

Eigenvalues: $\lambda_n = \mu_n^2 = (n\pi)^2$ for $n = 1,2,3,\ldots$

Eigenfunctions: $F_n(x) = C_n\sin(n\pi x)$

Now look at ② : $G''(t) + \lambda G(t) = 0$.

Let $G(t) = e^{st}$

$G'(t) = se^{st}$

$G''(t) = s^2 e^{st}$

Then

$G''(t) + \lambda G(t) = 0$

$s^2 e^{st} + \lambda e^{st} = 0$

$(s^2 + \lambda)e^{st} = 0$

$s^2 = -\lambda = -(n\pi)^2$

$s = \pm i(n\pi)$

$\therefore G_n(t) = a_n \sin(n\pi t) + b_n \cos(n\pi t)$ for $n = 1, 2, 3, \ldots$

$G_n'(t) = a_n n\pi \cos(n\pi t) - b_n n\pi \sin(n\pi t)$

(2)

Apply the IC.

$G_n'(0) = a_n n\pi = 0 \rightarrow a_n = 0$

$\therefore G_n(t) = b_n \cos(n\pi t)$ for $n = 1, 2, 3, \ldots$

Step 4: Apply superposition to obtain the general solution

$U_n(x,t) = F_n(x) G_n(t)$

$= B_n \sin(n\pi x) \cos(n\pi t)$ with $B_n = b_n C_n$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \cos(n\pi t)$$

(0.5)

Step 5: Apply the remaining IC

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = 2\sin(\pi x)$$

By inspection, we see that
$B_1 = 2$ and $B_n = 0 \ \forall\, n \neq 1$ (1)

Thus, the formal solution is

$$\boxed{u(x,t) = 2\sin(\pi x)\cos(\pi t)}$$ (0.5)

b) -The BCs $u(0,t) = u(1,t) = 0$ mean the ends of the string are held fixed for all time t (in other words,
(1) there is no vertical displacement of the string at the points $x=0$ and $x=1$).

-The IC $u(x,0) = 2\sin(\pi x)$ means that the initial
(1) position of the string is described by the sinusoidal curve $2\sin(\pi x)$.

-The IC $u_t(x,0) = 0$ means that the initial
(1) velocity of the string is 0 (in other words, the string is initially at rest).

6 pts
2) Find the general solution of $3u_x + 2u_y - 2u = 5x$.

Solution: *Note that if the question does not explicitly ask you to sketch the characteristic lines, you don't have to do this.
We use the method of characteristic lines.
We have $a=3$, and $b=2$, so the slope of the characteristic lines is $b/a = 2/3$.

$bx - ay = d$
$2x - 3y = d$.

1.5 Set up

We set $v(w,z) = u(x,y)$, where

$$\begin{cases} w = 2x - 3y \\ z = y \end{cases} \Longrightarrow \begin{cases} x = \frac{w+3z}{2} \\ y = z \end{cases}$$

The change of variables yields

$$3u_x + 2u_y = 3(v_w w_x + v_z z_x) + 2(v_w w_y + v_z z_y)$$
$$= 3(v_w \cdot 2 + v_z \cdot 0) + 2(v_w \cdot (-3) + v_z \cdot 1)$$
$$= 2v_z$$

So

$$3u_x + 2u_y - 2u = 5x \iff 2v_z - 2v = 5\left(\frac{w+3z}{2}\right)$$

Convert this ODE to standard form by dividing by 2

$$v_z - v = \frac{5}{4}(w+3z)$$

1.5

The integrating factor is

$$\mu = e^{\int -1 dz} = e^{-z}$$

Multiplying both sides of the above equation by the integrating factor gives

$$e^{-z}v_z - e^{-z}v = \frac{5}{4}(w+3z)e^{-z}$$
$$(e^{-z}v)_z = \frac{5}{4}we^{-z} + \frac{15}{4}ze^{-z}$$

Integrating both sides wrt z yields

$$\int (e^{-z}v)_z dz = \int \left(\frac{5}{4}we^{-z} + \frac{15}{4}ze^{-z}\right)dz$$

2

$$\int (e^{-z}v)_z dz = \frac{5}{4}w\int e^{-z}dz + \frac{15}{4}\int ze^{-z}dz$$

IBP: $u = z$ $\quad dv = e^{-z}dz$
$\quad du = dz$ $\quad v = -e^{-z}$

$$\int (e^{-z}v)_z dz = \frac{5}{4}w\int e^{-z}dz + \frac{15}{4}\left[-ze^{-z} - \int -e^{-z}dz\right]$$
$$e^{-z}v = \frac{-5}{4}we^{-z} + \frac{15}{4}\left(-ze^{-z} - e^{-z}\right) + c(w)$$

where $c(w)$ is an arbitrary function of w

$$V(w,z) = \frac{-5}{4}w - \frac{15}{4}(z+1) + c(w)e^{z}$$

The general solution in terms of x & y is given by

$$u(x,y) = \frac{-5}{4}(2x-3y) - \frac{15}{4}(y+1) + e^{y}F(2x-3y)$$

$$= \frac{-5}{4}(2x-\cancel{3y}+\cancel{3y}+3) + e^{y}F(2x-3y)$$

$$\boxed{u(x,y) = \frac{-5}{4}(2x+3) + e^{y}F(2x-3y)} \quad ①$$

where F is an arbitrary function of $2x-3y$.

[10 pts]

3) Find a formal solution to the given BVP.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on } 0<x<\pi \text{ and } 0<y<1,$$

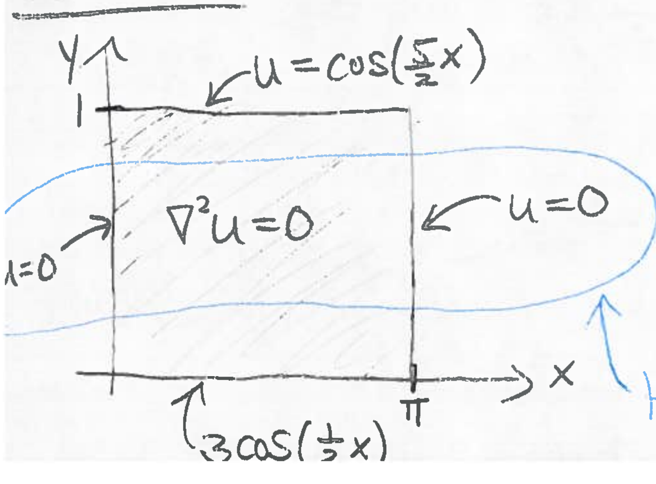
$$u_x(0,y) = u(\pi,y) = 0 \quad \text{for } 0<y<1,$$

$$u(x,0) = 3\cos\left(\tfrac{1}{2}x\right) \quad \text{for } 0<x<\pi,$$

$$u(x,1) = \cos\left(\tfrac{5}{2}x\right) \quad \text{for } 0<x<\pi.$$

When you go through and check the cases, cases 1 & 2 give the trivial solution, so do not check them. Only check the case where $\lambda > 0$.

Solution:



Homogeneous BCs in x

*Since the BCs in x are homogeneous, we can apply the method of separation of variables

4) Step 1:

Assume $u(x,y) = F(x)G(y)$

Substitute into the PDE.   ① Set up

$$F''(x)G(y) + F(x)G''(y) = 0$$

$$\frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} = 0$$

The F problem is the one we want to solve first, as the BCs in x are homogeneous. Rearrange the equation so that the F terms stay positive.

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda \qquad \lambda = \text{constant}$$

Step 2: Set up the 2 ODEs $F'(0) = F(\pi) = 0$ 1.5

① $F''(x) + \lambda F(x) = 0$

② $G''(y) - \lambda G(y) = 0$

We have homogeneous BCs in x:

$u_x(0,y) = F'(0)G(y) = 0 \qquad 0<y<1 \qquad \Rightarrow F'(0) = F(\pi) = 0.$

$u(\pi,y) = F(\pi)G(y) = 0$

Step 3: Solve the 2 ODEs.

* Solve ① first to find the eigenvalues.

Let $F(x) = e^{rx}$

$F'(x) = re^{rx}$

$F''(x) = r^2e^{rx}$

Then

$$r^2e^{rx} + \lambda e^{rx} = 0$$

$$(r^2+\lambda)e^{rx} = 0$$

$$r^2 + \lambda = 0$$

$$r^2 = -\lambda$$

*We are told that cases 1 & 2 yield the trivial solution, so we only need to check case 3.

<u>Case 3</u>: $\lambda > 0$, so set $\lambda = \mu^2$ with $\mu > 0$

Then $r^2 = -\mu^2 \rightarrow r = \pm i\mu$

$F(x) = C_3 \cos(\mu x) + C_4 \sin(\mu x)$

$F'(x) = -C_3 \mu \sin(\mu x) + C_4 \mu \cos(\mu x)$

$F'(0) = C_4 \mu = 0 \rightarrow C_4 = 0$

2

$\therefore F(x) = C_3 \cos(\mu x)$

$F(\pi) = C_3 \cos(\mu \pi) = 0$

$\cos(\mu\pi) = 0$

$\mu\pi = \left(\frac{2n-1}{2}\right)\pi$ for $n = 1, 2, 3, \ldots$

$\mu = \left(\frac{2n-1}{2}\right)$

Eigenvalues: $\lambda_n = (\mu_n)^2 = \left(\frac{2n-1}{2}\right)^2$ for $n = 1, 2, 3, \ldots$

Eigenfunctions: $F_n(x) = C_n \cos\left(\frac{2n-1}{2} x\right)$

*Now look at 2: $G''(y) - \lambda G(y) = 0$.

Let $G(y) = e^{sy}$

$G'(y) = s e^{sy}$

$G''(y) = s^2 e^{sy}$

Then

$s^2 e^{sy} - \lambda e^{sy} = 0$

$(s^2 - \lambda) e^{sy} = 0$

$s^2 - \lambda = 0$

$s^2 = \lambda = \left(\frac{2n-1}{2}\right)^2$

5)

$$S = \pm\left(\frac{2n-1}{2}\right)$$ Real values, so expect hyperbolic functions.

Apply the trick

Pick

$$G_n(y) = a_n \sinh\left(\left(\frac{2n-1}{2}\right) y\right) + b_n \sinh\left(\left(\frac{2n-1}{2}\right)(1-y)\right)$$ (1)

Step 4: Apply Superposition

$$U_n(x,y) = F_n(x)\,G_n(y)$$

$$= \cos\left(\left(\frac{2n-1}{2}\right)x\right)\left[A_n \sinh\left(\left(\frac{2n-1}{2}\right)y\right) + B_n \sinh\left(\left(\frac{2n-1}{2}\right)(1-y)\right)\right]$$

for $n = 1, 2, 3, \ldots$

where $A_n = a_n c_n$ and $B_n = b_n c_n$. (0.5)

The general solution is

$$u(x,y) = \sum_{n=1}^{\infty}\left[A_n \sinh\left(\left(\frac{2n-1}{2}\right)y\right) + B_n \sinh\left(\left(\frac{2n-1}{2}\right)(1-y)\right)\right]\cos\left(\left(\frac{2n-1}{2}\right)x\right)$$

Step 5: Apply the remaining BCs

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{2n-1}{2}\right)\cos\left(\left(\frac{2n-1}{2}\right)x\right) = 3\cos\left(\frac{1}{2}x\right)$$

By inspection we see that

When $n=1 \longrightarrow B_1 \sinh\left(\frac{1}{2}\right) = 3$ (1)

$$\Rightarrow B_1 = \frac{3}{\sinh(1/2)}$$

and $B_n = 0 \ \forall\ n \neq 1.$ (0.5)

$$u(x,1) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{2n-1}{2}\right) \cos\left(\frac{2n-1}{2}x\right) = \cos\left(\frac{5}{2}x\right)$$

By inspection, we see that

when $\frac{2n-1}{2} = \frac{5}{2} \Rightarrow n=3$

then $A_3 \sinh\left(\frac{5}{2}\right) = 1 \rightarrow A_3 = \frac{1}{\sinh(5/2)}$ 1

and

$A_n = 0 \quad \forall\, n \neq 3.$ 0.5

Thus, the formal solution is

$$u(x,0) = \frac{3}{\sinh(1/2)} \sinh\left(\frac{1}{2}(1-y)\right) \cos\left(\frac{1}{2}x\right) + \frac{1}{\sinh(5/2)} \sinh\left(\frac{5}{2}y\right) \cos\left(\frac{5}{2}x\right)$$

1

7/7.5

4) Obtain the Fourier sine series for the function

$$f(x) = \begin{cases} -1, & 0<x<1 \\ 2, & 1<x<3 \end{cases}$$

Discuss the convergence of this series, and show graphically the function represented by the series for all x.

Solution: Period $= 2L = 6 \Rightarrow L = 3$

*Start by constructing the odd extension.

Recall

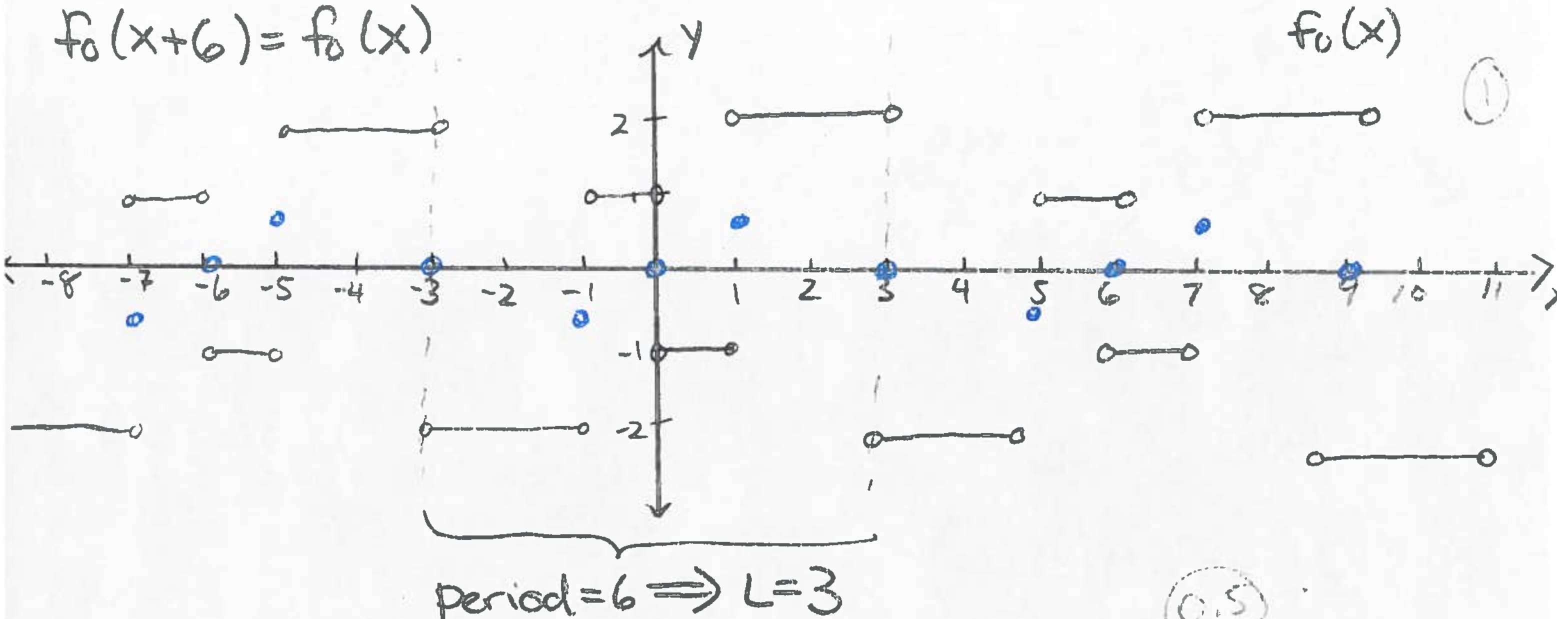
$$f_o(x) = \begin{cases} f(x), & 0<x<L \\ -f(-x), & -L<x<0 \end{cases}$$

So we have

6)

$$f_0(x)=\begin{cases}-2, & -3<x<-1\\ 1, & -1<x<0\\ -1, & 0<x<1\\ 2, & 1<x<3\end{cases} \qquad f_0'(x)=\begin{cases}0, & -3<x<-1\\ 0, & -1<x<0\\ 0, & 0<x<1\\ 0, & 1<x<3\end{cases}$$

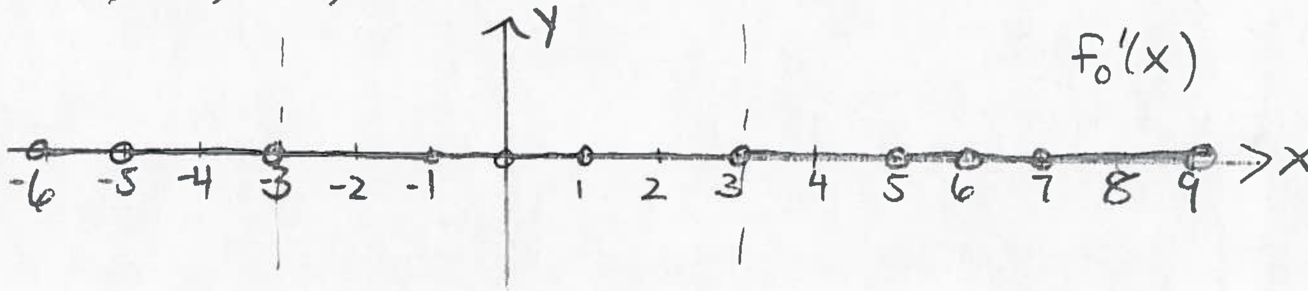
$f_0(x+6)=f_0(x)$



(1)

(0.5)

* f_0 is piecewise continuous on $[-3,3]$ with discontinuities at $0, \pm 3, \pm 6, \ldots$ and $\pm 1, \pm 5, \pm 9, \ldots$



(1)

* f_0' is piecewise continuous on $[-3,3]$. (0.5)

$\therefore$ Since f_0 & f_0' are piecewise continuous on $[-3,3]$, the series will converge pointwise to f_0 wherever it is continuous and at the points of discontinuity the series will converge to

(0.5)

$$\tfrac{1}{2}[f(x^+)+f(x^-)].$$

At $x=0$, $\tfrac{1}{2}[f(0^+)+f(0^-)]=\tfrac{1}{2}(-1+1)=0$ (1)

$x=1$, $\tfrac{1}{2}[f(1^+)+f(1^-)]=\tfrac{1}{2}(2+(-1))=\tfrac{1}{2}$

$x=-1, \quad \frac{1}{2}[f(-1^+) + f(-1^-)] = \frac{1}{2}(1+(-2)) = -\frac{1}{2}$

$x=\pm 3, \quad \frac{1}{2}[f(-3^+)+f(3^-)] = \frac{1}{2}(-2+2) = 0$

$f_0(x)$ converges to the 6-periodic function $g(x)$, where

$$g(x) = \begin{cases} -2, & -3<x<-1 \\ -\frac{1}{2}, & x=-1 \\ 1, & -1<x<0 \\ 0, & x=0,\pm 3 \\ -1, & 0<x<1 \\ \frac{1}{2}, & x=1 \\ 2, & 1<x<3 \end{cases}$$

* Compute the series

$$f_0(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \qquad L=3.$$

Integration ②

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\int_0^1 -1\cdot\sin\left(\frac{n\pi x}{3}\right)dx + \int_1^3 2\sin\left(\frac{n\pi x}{3}\right)dx\right]$$

$$= \frac{2}{3}\left[\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\Big|_0^1 - \frac{6}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\Big|_1^3\right]$$

$$= \frac{2}{3}\left[\frac{3}{n\pi}\left(\cos\left(\frac{n\pi}{3}\right)-1\right) - \frac{6}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{3}\right)\right)\right]$$

$$= \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{3}\right) - 1 - 2(-1)^n + 2\cos\left(\frac{n\pi}{3}\right)\right]$$

$$= \frac{2}{n\pi}\left[3\cos\left(\frac{n\pi}{3}\right) - 1 - 2(-1)^n\right]$$

7) Thus,

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[3\cos\left(\frac{n\pi}{3}\right) - 1 - 2(-1)^n\right] \sin\left(\frac{n\pi x}{3}\right)}$$ 0.5

12 pts

5) Consider the following BVP:

$$x^2 y'' + 3xy' + 26y = C + \frac{1}{x}$$
$$y(1) = y(e^{\pi}) = 0$$

a) State the Fredholm alternative.

b) Find the values of C for which the problem has a solution.

Solution:

a) Fredholm Alternative:

Let L be a linear differential operator and let B represent a set of linear BCs. The nonhomogeneous BVP

$$L[y](x) = h(x), \quad a < x < b,$$
$$B[y] = 0$$

has a solution iff 2

$$\int_a^b h(x) z(x) \, dx = 0$$

for every solution z of the adjoint BVP

$$L^+[z](x) = 0, \quad a < x < b,$$
$$B^+[z] = 0.$$

b) We are given

$$L[y]: x^2 y'' + 3xy' + 26y \qquad \text{with } h(x) = C + \frac{1}{x}$$
$$B[y]: y(1) = y(e^{\pi}) = 0$$

We have

$$A_2 = x^2, \quad A_1 = 3x, \quad A_0 = 26$$

The adjoint operator for L is

$$\begin{aligned}
L^+[y] &= (A_2 y)'' - (A_1 y)' + (A_0 y) \\
&= (x^2 y)'' - (3xy)' + 26y \qquad 2 \\
&= (2xy + x^2 y')' - (3y + 3xy') + 26y \\
&= (2y + 2xy' + 2xy' + x^2 y'') - 3y - 3xy' + 26y \\
&= x^2 y'' + xy' + 25y
\end{aligned}$$

To find the adjoint BCs, we need

$$P(u,v)\Big|_1^{e^\pi} = 0 \qquad \forall\ u \in D(L),\ v \in D(L^+)$$

with

$$\begin{aligned}
P(u,v) &= uA_1 v - u(A_2 v)' + u' A_2 v \\
&= u(3x)v - u(x^2 v)' + u' x^2 v \qquad 2 \\
&= 3xuv - u(2xv + x^2 v') + u' x^2 v \\
&= xuv - x^2(uv' - u'v)
\end{aligned}$$

$$P(u,v)\Big|_1^{e^\pi} = 0$$

$$\left[e^\pi \cancel{u(e^\pi)} v(e^\pi) + e^{2\pi}\left(u'(e^\pi)v(e^\pi) - \cancel{u(e^\pi)v'(e^\pi)}\right)\right] - \left[\cancel{u(1)v(1)} + \left(u'(1)v(1) - \cancel{u(1)v'(1)}\right)\right] = 0$$

Since $u \in D(L)$, we know

$$u(1) = u(e^\pi) = 0$$

$$\therefore\ e^{2\pi} u'(e^\pi)\, v(e^\pi) - u'(1)v(1) = 0$$

Because $u'(1)$ & $u'(e^\pi)$ are arbitrary, we must have

$$v(1) = v(e^\pi) = 0$$

3) Thus, the adjoint BVP is

$$L^+[y]: x^2y''+xy'+25y=0 \quad (1)$$

$$B^+[y]: y(1)=y(e^{\pi})=0.$$

Let's solve this problem. Set

$$y=x^r$$
$$y'=rx^{r-1}$$
$$y''=r(r-1)x^{r-2}$$

Then

$$x^2r(r-1)x^{r-2}+xrx^{r-1}+25x^r=0$$
$$(r^2-r+r+25)x^r=0$$
$$r^2+25=0 \quad (0.5)$$
$$r^2=-25$$
$$r=\pm 5i$$

The general solution is (0.5)

$$y(x)=C_1\cos(5\ln x)+C_2\sin(5\ln x)$$

$$y(1)=C_1=0 \longrightarrow y(x)=C_2\sin(5\ln x)$$

$$y(e^{\pi})=C_2\sin(5\pi)=0 \checkmark$$ Trivially satisfied

$\therefore$ Every solution to the adjoint problem has the form

$$y(x)=C_2\sin(5\ln x) \quad (1)$$

where C_2 is free.

*By the Fredholm alternative, the nonhomogeneous problem has a solution iff

$$\int_1^{e^{\pi}}\left(C+\frac{1}{x}\right)\sin(5\ln x)\,dx=0 \quad (0.5)$$

$$C\int_1^{e^\pi} \sin(5\ln x)\,dx + \int_1^{e^\pi} \frac{1}{x}\sin(5\ln x)\,dx = 0$$

$u = \ln x \rightarrow e^u = x$ $\qquad x = 1 \rightarrow u = 0$

$du = \frac{1}{x}dx \rightarrow dx = e^u du$ $\qquad x = e^\pi \rightarrow u = \pi$

$$C\underbrace{\int_0^\pi e^u \sin(5u)\,du}_{①} + \underbrace{\int_0^\pi \sin(5u)\,du}_{②} = 0$$

$$② = -\frac{\cos(5u)}{5}\Big|_0^\pi = -\frac{1}{5}(\cos(5\pi) - 1) = \frac{2}{5}$$

By the useful integral at the start of the exam, we know

$$① = \frac{e^u}{5^2+1}(\sin(5u) - 5\cos(5u))\Big|_0^\pi$$

$$= \frac{1}{26}\left[e^\pi(\sin(5\pi)^{\to 0} - 5\cos(5\pi)) - (0 - 5\cdot 1)\right]$$

$$= \frac{5}{26}(e^\pi + 1)$$

$$\therefore\ C\left(\frac{5}{26}(e^\pi+1)\right) + \frac{2}{5} = 0$$

$$C = -\frac{2}{5}\cdot\frac{26}{5(e^\pi+1)}$$

$$= \boxed{\frac{-52}{25(e^\pi+1)}}$$

2

(9) Thus, by the Fredholm alternative

$$x^2 y'' + 3x y' + 26 y = \frac{-52}{25(e^{\pi}+1)} + \frac{1}{x} \qquad (0.5)$$

$$y(1) = y(e^{\pi}) = 0$$

has a solution.

6pts

6) Let L be the differential operator defined by

$$L[y] := A_2(x) y''(x) + A_1(x) y'(x) + A_0(x) y(x).$$

Start with the inner product $(L[u], v)$ and use integration by parts to obtain formulas for the formal adjoint $L^+[y]$ and the bilinear concomitant $P(u,v)$ defined by Green's formula:

$$\int_a^b (L[u]v - uL^+[v])\,dx = P(u,v)(x)\Big|_a^b.$$

Solution:

By the definition of the inner product, we can write

$$(L[u], v) = \int_a^b (A_2 u'' + A_1 u' + A_0 u) v\,dx$$

$$= \int_a^b (A_2 u'' v + A_1 u' v + A_0 u v)\,dx$$

$$= \underbrace{\int_a^b A_2 u'' v\,dx}_{①} + \underbrace{\int_a^b A_1 u' v\,dx}_{②} + \int_a^b A_0 u v\,dx \qquad (1)$$

① IBP: $u_1 = A_2 v \qquad dv_1 = u''dx$

$du_1 = (A_2 v)'dx \qquad v_1 = u'$

$$① = u' A_2 v \Big|_a^b - \int_a^b u'(A_2 v')\,dx$$

IBP: $u_1 = (A_2 v)'$ $\qquad dv_1 = u'dx$

$du_1 = (A_2 v)''dx \qquad v_1 = u$

$$① = u'A_2 v\Big|_a^b - \left[u(A_2 v)'\Big|_a^b - \int_a^b u(A_2 v)''dx \right] \quad ②$$

② IBP: $u_2 = A_1 v \qquad dv_2 = u'$

$du_2 = (A_1 v)'dx \qquad v_2 = u$

$$② = uA_1 v\Big|_a^b - \int_a^b u(A_1 v)'dx \quad ①$$

$$\therefore (L[u], v) = \left[u'A_2 v\Big|_a^b - u(A_2 v)'\Big|_a^b + \int_a^b u(A_2 v)''dx \right] + \left[uA_1 v\Big|_a^b - \int_a^b u(A_1 v)'dx \right] + \int_a^b uA_0 v\,dx$$

$$= \int_a^b u \underbrace{\left((A_2 v)'' - (A_1 v)' + A_0 v \right)}_{L^+[v] \text{ (formal adjoint)}} dx + \underbrace{\left[uA_1 v - u(A_2 v)' + u'A_2 v \right]}_{\substack{P(u,v) \\ \text{(bilinear concomitant} \\ \text{associated with } L)}}\Bigg|_a^b$$

②

$$= (u, L^+[v]) + P(u,v)(x)\Big|_a^b$$

$$\Longleftrightarrow \int_a^b \left(L[u]v - uL^+[v] \right) dx = P(u,v)(x)\Big|_a^b$$

$$\therefore L^+[v] = (A_2 v)'' - (A_1 v)' + A_0 v$$

$$P(u,v) = uA_1 v - u(A_2 v)' + u'A_2 v$$

10) 7 pts

7) Consider the following BVP

$$y'' - 2y + \lambda y = 0, \quad 0 < x < \pi,$$
$$y(0) = y'(\pi) = 0.$$

a) Convert this equation into a Sturm-Liouville equation and verify that the problem is regular.

b) Use the fact that the eigenvalues and eigenfunctions for this BVP are given by

$$\lambda_n = 2 + \left(\frac{2n-1}{2}\right)^2 \quad \text{for } n = 1, 2, 3, \ldots$$

and

$$y_n(x) = C_n \sin\left(\frac{2n-1}{2} x\right)$$

to express the function $f(x) = 1$ in an eigenfunction expansion on $0 < x < \pi$.

Solution:

a) By inspection, it is clear that the Sturm-Liouville form for the equation is

$$(y')' - 2y + \lambda y = 0.$$

3

*Our BVP is regular since

① $p(x) = 1$, $p'(x) = 0$, $q(x) = -2$, and $r(x) = 1$ are real-valued continuous functions on $[0, \pi]$.

② $p(x) = 1 > 0$ and $r(x) = 1 > 0$ on $[0, \pi]$.

**Make sure this is the closed interval. Many of you used round brackets on the midterm when you checked regularity

b) Begin by normalizing the eigenfunctions. We need to find C_n such that

$$\int_0^{\pi} y_n^2(x)\, r(x)\, dx = 1$$

$$\int_0^{\pi} C_n^2 \sin^2\left(\frac{2n-1}{2}x\right)\cdot 1\, dx = 1$$

$$C_n^2 \int_0^{\pi} \frac{1-\cos((2n-1)x)}{2}\, dx = 1$$

$$\frac{C_n^2}{2}\left[x - \frac{\sin((2n-1)x)}{2n-1}\right]\Bigg|_0^{\pi} = 1$$

$$\frac{C_n^2}{2}\left[\pi - \cancel{\frac{\sin((2n-1)\pi)}{2n-1}}^{\,0} - 0\right] = 1 \qquad 2$$

$$\frac{C_n^2 \pi}{2} = 1$$

$$C_n^2 = 2/\pi$$

$$C_n = \sqrt{2/\pi}$$

* The normalized eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin\left(\frac{2n-1}{2}x\right) \qquad \text{for } n=1,2,3,\ldots$$

*We want to express $f(x)=1$ in an eigenfunction expansion.

We know

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{where} \qquad a_n = \int_a^b f(x)\phi_n(x) r(x)\, dx,$$

So we can write

$$1 = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{\pi}}\sin\left(\frac{2n-1}{2}x\right)$$

where

$$a_n = \int_0^{\pi} 1\cdot\sqrt{\frac{2}{\pi}}\sin\left(\frac{2n-1}{2}x\right) dx$$

11)

$$a_n = -\sqrt{\frac{2}{\pi}}\frac{2}{2n-1}\cos\left(\frac{2n-1}{2}x\right)\Big|_0^{\pi}$$

$$= \frac{-2}{2n-1}\sqrt{\frac{2}{\pi}}\left(\cos\left(\frac{2n-1}{2}\pi\right) - 1\right)$$

$$= \frac{2}{2n-1}\sqrt{\frac{2}{\pi}}$$ (1)

Thus, the eigenfunction expansion for $f(x)=1$ is

$$1 = \sum_{n=1}^{\infty} \frac{2}{2n-1}\sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\sin\left(\frac{2n-1}{2}x\right)$$

$$1 = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}x\right).$$ (1)