A#5- Sol'ns

1. Verify that the eigenfunctions of

$$y'' + 2y' + (1+\lambda^2)y = 0, \quad 0 < x < 1 \quad \ldots (1)$$

$$y(0) = y'(1) = 0 \quad \ldots (2)$$

(a) form an orthogonal set. Then (b) express $f(x) = 1$ in an eigenfunction expansion.

a) The ODE (1) is linear with constant coefficients, so we set

$$y = e^{rx} \Rightarrow y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

We arrive at the characteristic equation

$$r^2 + 2r + (1+\lambda^2) = 0 \Leftrightarrow r = -1 \pm \sqrt{1-(1+\lambda^2)} = -1 \pm \lambda i$$

Case 1: $\lambda = 0$ $\Rightarrow r = -1$ a double root

$$\therefore y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

BCs: $\begin{cases} y(0) = 0 \\ y'(1) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 e^{-1} - c_2 e^{-1} = 0 \end{cases}$ trivially satisfied

$\therefore$ $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y_0(x) = xe^{-x}$

Case 2: $\lambda \neq 0$

The general solution is

$$y(x) = e^{-x}\left(c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\right)$$

BCs:

$$y(0) = 0 \Leftrightarrow c_1 = 0$$

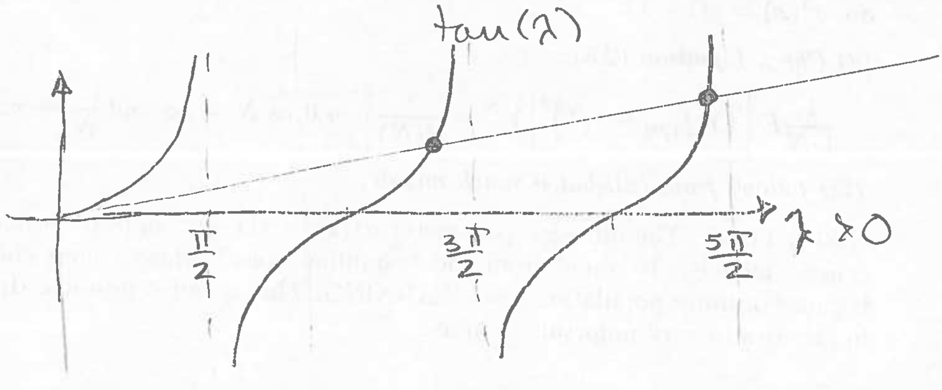
$$y'(1) = 0 \Leftrightarrow \left[-e^{-x} c_2 \sin(\lambda x) + e^{-x} \lambda c_2 \cos(\lambda x)\right]_{x=1}$$

$$\Leftrightarrow c_2 e^{-1}\left(\lambda \cos(\lambda) - \sin(\lambda)\right) = 0$$

For nontrivial solutions we require $c_2 \neq 0$, and so, $\because e^{-x} \neq 0$, we must have

$$\lambda \cos(\lambda) - \sin(\lambda) = 0 \Leftrightarrow \boxed{\tan(\lambda) = \lambda} \quad \ldots (3)$$

Equation (3) is a transcendental equation for λ, so we plot the solutions:



As $n \to \infty$, $\lambda_n \to \dfrac{(2n-1)\pi}{2}$.

We see that there are multiple intersections $\lambda_n > 0$ satisfying (3). For these eigenvalues, the eigenfunctions are

$$y_n(x) = e^{-x} \sin(\lambda_n x)$$

Thus, the full set of eigenfunctions is

$$\left\{ x e^{-x},\ e^{-x} \sin(\lambda_n x) \right\} \quad \ldots\ldots (4)$$

where the λ_n are given by (3).

Now we need to verify <u>orthogonality</u> of the set (4). That is, we require

$$\int_0^1 \phi_n(x)\, \phi_m(x)\, r(x)\, dx = \begin{cases} 0 & n \neq m \\ \neq 0 & n = m \end{cases} \quad \ldots\ldots (5)$$

where, in our case, the eigenfunctions $\phi_n(x)$ are given by (4), and $r(x)$ is found by putting (1) into S-L form:

$A_0(x) = 1+\lambda^2, \quad A_1(x) = 2, \quad A_2(x) = 1$

$$\mu(x) = \frac{1}{A_2(x)} e^{\int \frac{A_1(x)}{A_2(x)} dx} = e^{\int 2 dx} = e^{2x}$$

Multiplying (1) by $\mu(x)$ we obtain

$$e^{2x} y'' + 2e^{2x} y' + e^{2x} y + \lambda^2 e^{2x} y = 0 \iff$$

$$\iff (e^{2x} y')' + e^{2x} y + \xi e^{2x} y = 0 \quad \ldots (6)$$

where $\xi = \lambda^2$. Equation (6) is in S-L form where $p(x) = e^{2x}$, $q(x) = e^{2x}$, $r(x) = e^{2x}$.

We now verify the orthogonality condition (5). This will take several steps:

① $$\int_0^1 \phi_0 \phi_n e^{2x} dx = \int_0^1 x e^{-x} e^{-x} \sin(\lambda_n x) e^{2x} dx$$
$n \neq 0$

$$= \int_0^1 x \sin(\lambda_n x) dx$$

$$= \left[-x \frac{\cos(\lambda_n x)}{\lambda_n} + \frac{\sin(\lambda_n x)}{\lambda_n^2} \right]_0^1 = \text{III}$$

$$\boxed{III} = \left[\frac{-\cos(\lambda_n)}{\lambda_n} + \frac{\sin(\lambda_n)}{\lambda_n^2} \right] = \frac{\cos(\lambda_n)}{\lambda_n} \left[-1 + \frac{\tan(\lambda_n)}{\lambda_n} \right]$$

$$= \frac{\cos(\lambda_n)}{\lambda_n} \left[-1 + \frac{\lambda_n}{\lambda_n} \right] \text{ from (3)}$$

$$= 0 \text{ as required}$$

(2) $$\int_0^1 \phi_n(x)\phi_m(x)\, e^{2x}\, dx = \int_0^1 e^{-x} \sin(\lambda_n x)\, e^{-x} \sin(\lambda_m x)\, e^{2x}\, dx$$

$n \neq m \neq 0$

$$= \int_0^1 \sin(\lambda_n x) \sin(\lambda_m x)\, dx$$

$$= \int_0^1 \frac{\cos((\lambda_n - \lambda_m)x) - \cos((\lambda_n + \lambda_m)x)}{2}\, dx$$

$$= \frac{1}{2} \left[\frac{\sin((\lambda_n - \lambda_m)x)}{\lambda_n - \lambda_m} - \frac{\sin((\lambda_n + \lambda_m)x)}{\lambda_n + \lambda_m} \right.$$

$$= \frac{1}{2} \left[\frac{\sin(\lambda_n - \lambda_m)}{\lambda_n - \lambda_m} - \frac{\sin(\lambda_n + \lambda_m)}{\lambda_n + \lambda_m} \right]$$

$$= \boxed{IV}$$

(6)

$$\underline{IV} = \frac{1}{2}\left[\frac{\sin(\lambda_n)\cos(\lambda_m) - \sin(\lambda_m)\cos(\lambda_n)}{\lambda_n - \lambda_m} - \frac{\sin(\lambda_n)\cos(\lambda_m) + \sin(\lambda_m)\cos(\lambda_n)}{\lambda_n + \lambda_m}\right]$$

$$= \frac{1}{2}\left[\sin(\lambda_n)\cos(\lambda_m)\left(\frac{1}{\lambda_n - \lambda_m} - \frac{1}{\lambda_n + \lambda_m}\right) - \sin(\lambda_m)\cos(\lambda_n)\left(\frac{1}{\lambda_n - \lambda_m} + \frac{1}{\lambda_n + \lambda_m}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2\lambda_m \sin(\lambda_n)\cos(\lambda_m) - 2\lambda_n \sin(\lambda_m)\cos(\lambda_n)}{\ldots}\right]$$

$$= \tan(\lambda_m)\sin(\lambda_n)\cos(\lambda_m) - \tan(\lambda_n)\sin(\lambda_m)\cos(\lambda_n)$$

(by (3))

$$= \sin(\lambda_m)\sin(\lambda_n) - \sin(\lambda_n)\sin(\lambda_m)$$

$= 0$ as required.

(3) $$\int_0^1 \phi_0^2(x)\, e^{2x}\, dx = \int_0^1 x^2 e^{-2x} e^{2x}\, dx = \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$\neq 0$ as required

4 $$\int_0^1 \phi_n^2(x) e^{2x}\,dx = \int_0^1 e^{-2x} \sin^2(\lambda_n x) e^{2x}\,dx$$

$$= \int_0^1 \sin^2(\lambda_n x)\,dx$$

$$= \int_0^1 \frac{1-\cos(2\lambda_n x)}{2}\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2\lambda_n}\sin(2\lambda_n x)\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{4\lambda_n}\sin(2\lambda_n) \quad = I$$

We show that $I \neq 0$ by contradiction. Assume $I = 0$. Then

$$\frac{1}{2} - \frac{1}{4\lambda_n}\sin(2\lambda_n) = 0 \iff \sin(2\lambda_n) = \frac{4\lambda_n}{2} = 2\lambda_n$$

$$\iff \sin(2\lambda_n) = 2\tan(\lambda_n) \quad \text{(from (3))}$$

$$\iff \sin(2\lambda_n) = 2\,\frac{\sin(\lambda_n)}{\cos(\lambda_n)}$$

$$\iff \cos(\lambda_n) = 2$$

which is impossible. So $I \neq 0$.

(There is no page 8. The elves swallowed it. :))

Now write f(x) as an eigenfunction expansion.

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n \phi_n(x) \iff f(x) = \sum_{n=1}^{\infty} \gamma_n \phi_n(x)\, r(x) \quad \ldots (9)$$

We need the r(x) for orthogonality. Then

$$\gamma_n = \frac{\int_0^1 f(x)\, \phi_n(x)\, dx}{\int_0^1 \phi_n^2(x)\, r(x)\, dx} \quad \ldots\ldots (10)$$

$$\therefore \quad \gamma_0 = \frac{\int_0^1 1 \cdot x e^{-x} \cdot dx}{\frac{1}{3}} \quad \text{(from calculation 3 on p 6)}$$

$$= 3\left[(-x-1)e^{-x}\right]_0^1 = -3\left[2e^{-1} + e^{-1}\right] = -9e^{-1}$$

(calculations continue on the next p)

$$\gamma_n = \frac{\int_0^1 1 \cdot e^{-x} \sin(\lambda_n x)\, dx}{\frac{1}{2} - \frac{1}{4\lambda_n}\sin(2\lambda_n)}$$

$n \in \mathbb{N}$

by calculation 4 on p 7

$$= \frac{1}{\frac{1}{2}\left[1 - \frac{1}{\cos(2\lambda_n)}\right]} \left[\frac{1}{\lambda_n^2+1} e^{-x} \left(-\sin(\lambda_n x) - \lambda_n \cos(\lambda_n x \right.\right.$$

(by (3))

$$= \frac{-1}{\frac{1}{2}\left(1 - \frac{1}{\cos(2\lambda_n)}\right)\left(\lambda_n^2+1\right)} \left[e^{-1}\left(\sin(\lambda_n) + \lambda_n \cos(\lambda_n)\right) - \lambda_n \right]$$

$$= \frac{-1}{\frac{1}{2}\left(1 - \frac{1}{\cos(2\lambda_n)}\right)\left(\lambda_n^2+1\right)} \left[e^{-1}\cos(\lambda_n)\left(\lambda_n + \lambda_n\right) - \lambda_n \right]$$

$$= -\frac{\lambda_n \left(e^{-1} 2\cos(\lambda_n) - 1 \right)}{\frac{1}{2}\left(1 - \frac{1}{\cos(2\lambda_n)}\right)\left(\lambda_n^2 + 1\right)}$$

$$\therefore\ f(x) = -9e^{-1}xe^{-x} + \sum_{n=1}^{\infty} \frac{2\lambda_n\left(2e^{-1}\cos(\lambda_n) - 1\right)}{\left(1 - \frac{1}{\cos(2\lambda_n)}\right)(\lambda_n^2 + 1)} e^{-x}\sin(\lambda_n x)$$

$n \in \mathbb{N}$.

2) Show that the problem

$$x^2y'' - \lambda(xy' - y) = 0 \qquad \leftarrow \text{Cauchy type equation}$$

$$y(1) = 0, \qquad 1 < x < 2 \qquad x^2y'' - \lambda xy' + \lambda y = 0$$

$$y(2) - y'(2) = 0,$$

here p

has only one real eigenvalue, and find the corresponding eigenfunction

SOLUTION: - Set $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$x^2 \cdot r(r-1)x^{r-2} - \lambda x r x^{r-1} + \lambda x^r = 0 \qquad xy$$

$$x^r \left(\underbrace{r(r-1) - \lambda r + \lambda}\right) = 0$$

$1 < x < 2$
$\neq 0$.

$$r^2 - r - \lambda r + \lambda = 0$$

$$r^2 - (1+\lambda)r + \lambda = 0$$

roots are

$$\lambda^2 + 2\lambda + 1 - 4\lambda = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$$

$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$r = \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda}}{2} = \frac{(1+\lambda) \pm (\lambda - 1)}{2}$$

$$= \frac{1+\lambda-\lambda+1}{2} = 1 \quad \text{or} \quad \frac{1+\lambda+\lambda-1}{2} = \lambda$$

So we must consider 2 cases $\lambda = 1$ double root, $\lambda \neq 1$ ...

Case 1. $\lambda = 1$

We have a double root $\Rightarrow$ general solution is

$$y(x) = c_1\cancel{x} + c_2 x \ln x \quad \to \quad y'(x) = c_2 \ln x + c_2 x \frac{1}{x}$$
$$= c_2(\ln x + 1)$$

apply BC $y(1) = 0 = c_1$

$$y(2) - y'(2) = 0 = c_2(2\ln 2 - \ln 2 - 1) = c_2(\ln 2 - 1)$$

$\Rightarrow c_2 = 0$ so $\lambda = 1$ is not an ...

case 2 : $\lambda \neq 1$

the roots are $r = 1$ and $r = \lambda$

general solution is $y(x) = c_1 x + c_2 x^{\lambda}$

Apply B.C

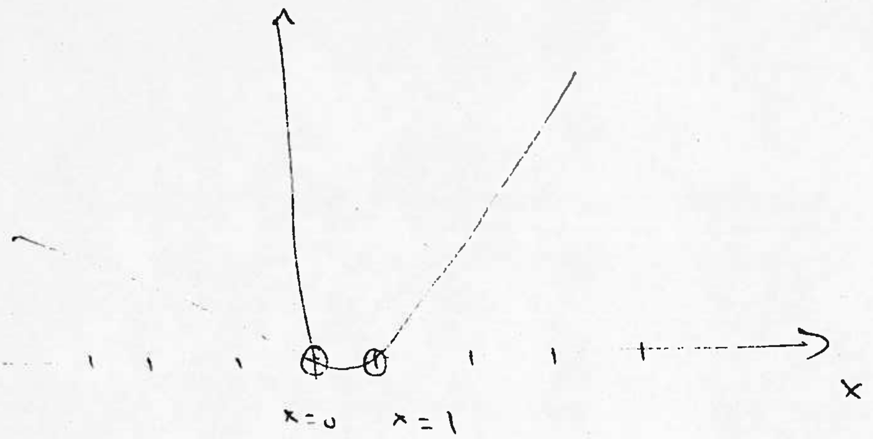
$y(1) = 0 \Rightarrow c_1 + c_2 = 0 \quad c_2 = -c_1$

so $y(x) = c_1 (x - x^{\lambda})$, $y'(x) = c_1 (1 - \lambda x^{\lambda - 1})$

$y(2) - y'(2) = 0 \Rightarrow c_1 (2 - 2^{\lambda} - 1 + \lambda 2^{\lambda - 1}) = 0$

$(1 - 2^{\lambda} + \lambda 2^{\lambda - 1}) = 0$

$\hookrightarrow$ look for roots of $f(x) = 1 - 2^{x} + x 2^{x-1}$



$\hookrightarrow$ this means the roots are $\lambda = 0$ or $\lambda = 1$

but $\lambda = 1$ is not an eigenvalue.

so $\lambda = 0$ is the only eigenvalue

the corresponding eigenfunction is $y(x) = x - 1$

~~Thus,~~

$$\cancel{\int_0^{\pi}(e^{-x}\sin(nx))(e^{-x}\sin(mx))\cdot e^{2x}\,dx \begin{cases} =0 & \text{if } n\neq m \\ =\pi/2\neq 0 & \text{if } n=m\end{cases}}$$

~~So~~

$$\cancel{\{e^{-x}\sin(nx)\}_{n=1}^{\infty}}$$

~~is an orthogonal set of eigenfunctions.~~

3) Consider the following BVP

$$x^2y''+2xy'+\frac{5}{4}y=h(x), \qquad 1<x<e^{\pi}$$

$$y(1)=y(e^{\pi})=0.$$

a) Find the adjoint BVP for the associated homogeneous problem.

b) Determine the conditions on $h(x)$ that guarantee that the given nonhomogeneous BVP has a solution

Solution:

a) We are given

$L[y]: x^2y''+2xy'+\frac{5}{4}y$

$B[y]: y(1)=y(e^{\pi})=0$

We have

$A_2=x^2,\ A_1=2x,\ A_0=5/4$

The adjoint operator for L is

$$\begin{aligned}L^+[y]&=(A_2y)''-(A_1y)'+A_0y\\&=(x^2y)''-(2xy)'+5/4\,y\\&=(2xy+x^2y')'-(2y+2xy')+5/4\,y\end{aligned}$$

$$L^+[y] = 2y + 2xy' + 2xy' + x^2y'' - 2y - 2xy' + 5/4\, y$$
$$= x^2y'' + 2xy' + 5/4\, y = 0$$

To find the adjoint BCs, we need

$P(u,v)(x)\Big|_0^{e^\pi} = 0$ where $u \in D(L)$, $v \in D(L^+)$

and

$$P(u,v) = uA_1v - u(A_2v)' + u'A_2v$$
$$= u(2x)v - u(x^2v)' + u'(x^2)v$$
$$= 2xuv - u(2xv + x^2v') + u'x^2v$$
$$= x^2(u'v - uv')$$

$$P(u,v)(x)\Big|_1^{e^\pi} = 0$$

$$\left[e^{2\pi}(u'(e^\pi)v(e^\pi) - u(e^\pi)v'(e^\pi)) - 1^2(u'(1)v(1) - u(1)v'(1))\right] = 0$$

Since $u \in D(L)$, we have

$$u(1) = u(e^\pi) = 0$$

$$\therefore e^{2\pi}u'(e^\pi)v(e^\pi) - u'(1)v(1) = 0$$

Because $u'(e^\pi)$ and $u'(1)$ are arbitrary, we must have

$$v(1) = v(e^\pi) = 0$$

Thus, the adjoint BVP is

$L^+[y]: x^2y'' + 2xy' + 5/4\, y = 0$

$B^+[y]: y(1) = y(e^\pi) = 0.$

4 b) In order to determine the conditions on $h(x)$ that guarantee that the given nonhomogeneous BVP has a solution, we must apply the Fredholm alternative. Therefore, we need to solve the adjoint BVP: (11

$$L^+[y] : x^2y'' + 2xy' + 5/4\, y = 0$$
$$B^+[y] : y(1) = y(e^{\pi}) = 0.$$

This is a Cauchy-Euler problem, so set

$$y(x) = x^r$$
$$y'(x) = r x^{r-1}$$
$$y''(x) = r(r-1)x^{r-2}$$

Then

$$x^2 r(r-1)x^{r-2} + 2x r x^{r-1} + 5/4\, x^r = 0$$
$$(r^2 - r + 2r + 5/4)x^r = 0$$
$$r^2 + r + 5/4 = 0$$
$$r = \frac{-1 \pm \sqrt{1-4(5/4)}}{2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = \frac{-1}{2} \pm i$$

The general solution is

$$y(x) = c_1 x^{-1/2}\cos(\ln x) + c_2 x^{-1/2}\sin(\ln x)$$
$$y(1) = c_1 \cdot 1 + c_2 \cdot 0 = 0 \rightarrow c_1 = 0$$
$$\therefore y(x) = c_2 x^{-1/2}\sin(\ln x)$$

$$y(e^{\pi}) = c_2 e^{-\pi/2}\sin(\ln e^{\pi}) = 0 \checkmark$$ Trivially satisfied

$\therefore$ Every solution to the adjoint problem has the form

$$y(x) = c_2 x^{-1/2}\sin(\ln x)$$ where c_2 is free

By the Fredholm Alternative, the nonhomogeneous problem has a solution iff (16)

$$\int_1^{e^\pi} h(x)\sqrt{x}\,\sin(\ln(x))\,dx = 0.$$

4) $L[y] = (1+x^3)y'' + 3x^2y' + \lambda r y$

$L^+[y] = [(1+x^3)y]'' - [3x^2 y]' + \lambda r y$

$= \left(3x^2y + (1+x^3)y'\right)' - (6xy + 3x^2y')$

$+ \lambda r y$

$= \cancel{6xy} + \cancel{3x^2y'} + 3x^2y' + (1+x^3)y''$

$- \cancel{6xy} - \cancel{3x^2y'} + \lambda r y$

$= (1+x^3)y'' + 3x^2y' + \lambda r y$

$\therefore L[y] = L^+[y]$.

Now consider the BCs. We require

$$P(u,v)\Big|_0^\pi = 0$$

for u in $D(L)$ & v in $D(L^+)$.

with $A_1 = 3x^2$ & $A_2 = (1+x^3)$ we have

$$P(u,v)\Big|_0^{\pi} = 0 \iff \left[u\,3x^2 v - u\big((1+x^3)v\big)' + u'(1+x^3)v\right]_0^{\pi} = 0$$

$$\iff \cancel{u(\pi)3\pi^2 v(\pi)} - u(\pi)\left[\cancel{3\pi^2 v} + (1+\pi^3)v'(\pi)\right]$$

$$+ u'(\pi)(1+\pi^3)v(\pi)$$

$$-\Big(\cancel{u(0)\cdot 0\cdot v(0)}^{\to 0} - u(0)(v'(0))$$

$$+\cancel{u'(0)(1)v(0)}^{\to 0}\Big) = 0$$

$$\iff -u(\pi)(1+\pi^3)v'(\pi) + u'(\pi)(1+\pi^3)v(\pi)$$

$$+u(0)v'(0) = 0$$

$$\iff (1+\pi^3)\left[u'(\pi)v(\pi) - u(\pi)v'(\pi)\right]$$

$$+u(0)v'(0) = 0$$

$$\iff (1+\pi^3)\left[u'(\pi)v(\pi) - (1+\pi^3)u'(\pi)v'(\pi)\right]$$

$$+u(0)v'(0) = 0$$

$$\iff (1+\pi^3)\,u'(\pi)\left[v(\pi) - (1+\pi^3)v'(\pi)\right] = 0$$

$$+u(0)v'(0) = 0$$

$\because$ $u'(\pi)$ & $u(0)$ are arbitrary, we must have

$$\begin{cases} v'(0) = 0 \\ v(\pi) - (1+\pi^3)v'(\pi) = 0 \end{cases}$$

These are the adjoint BCs, B^+.

$\because L = L^+$, & $B = B^+$, the BVP is selfadjoint.