a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 319
Date: Nov 6th, 2014 Time: 12:30pm Duration: 45 minutes.
This exam has 7 questions for a total of 40 points.

NAME: Solutions

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

This is a two-stage exam. You have 45 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Problem Number	1	2	3	4	5	6	7	Total
Points Earned								
Points Out Of	6	3	6	6	6	5	~~4~~ 5	~~40~~ 35

+ 5 bonus

- not included in indiv. te
- bonus on group test.

[6] 1. Consider the PDE problem below:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,$$
$$u(0,t) = U_1, \quad \frac{\partial u}{\partial x}(L,t) = U_2, \qquad t > 0$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

Note that U_1 and U_2 are constants. Derive and solve the steady state problem that arises from the PDE problem above.

$u(x,t) = v(x) + w(x,t)$ where $\lim_{t\to\infty} w(x,t) = 0$

Then, for large t the PDE becomes

$\beta v'' = 0 \iff v = Ax + B$

Also, for large t the BCs become

$$\begin{cases} \lim_{t\to\infty}\left(v(0) + w(0,t)\right) = v(0) = U_1 \\ \lim_{t\to\infty}\left(v'(L) + \frac{\partial w}{\partial x}(L,t)\right) = v'(L) = U_2 \end{cases} \iff \begin{cases} B = U_1 \\ A = U_2 \end{cases}$$

$\therefore \boxed{v(x) = U_2 x + U_1}$

[3] 2. Given that

$$\frac{\partial^2 u}{\partial t^2} = 25 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u_t(x,0) = 0, \quad u(x,0) = f(x), \qquad -\infty < x < \infty,$$

what is the d'Alembert solution? Write it out algebraically, and then show what this solution means on the axis below by sketching the solution at t=2.

$u(x,t) = \frac{1}{2}\left(f(x+5t) + f(x-5t)\right)$

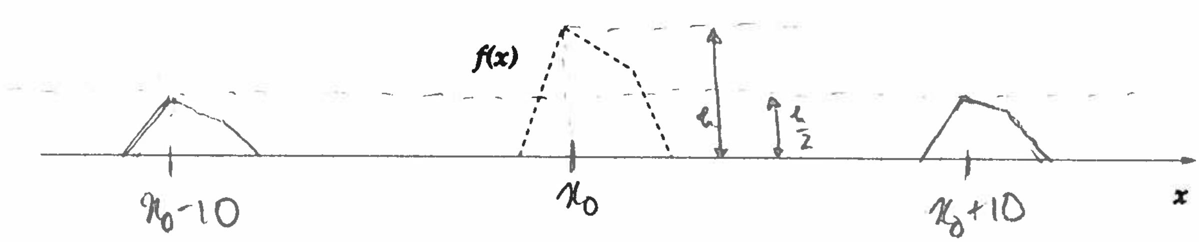


Figure 1: Plot of $f(x)$ for question 2.

$\boxed{6}$ 3. Consider the BVP below:

$$F''(x) + \mu^2 F(x) = 0, \qquad F'(0) = 0, \quad F(1) + F'(1) = 0.$$

Show that this BVP has multiple solutions (assume $\mu > 0$).

The general solution is

$$F(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

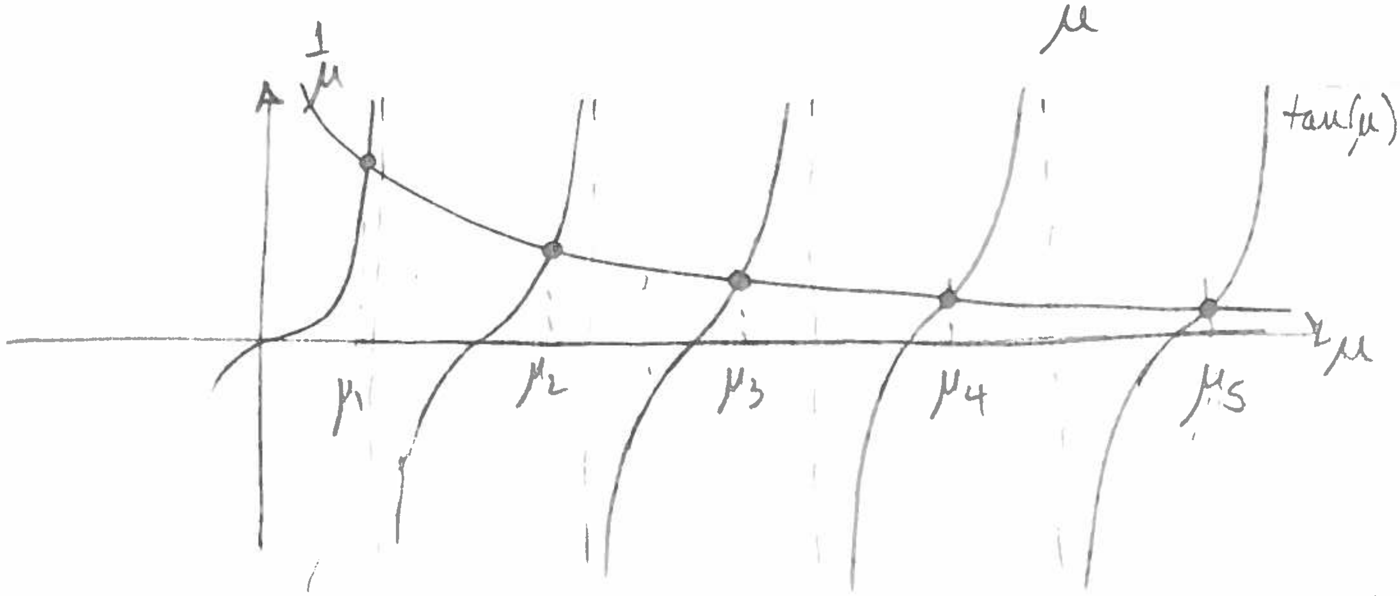
Apply the BCs

$$F'(0) = 0 \iff \underbrace{-\mu c_1 \sin(\mu x)\Big|_{x=0}}_{\to 0} + \mu c_2 \cos(\mu x)\Big|_{x=0} = 0 \iff c_2 = 0$$

$$F(1) + F'(1) = 0 \iff c_1 \cos(\mu) - \mu c_1 \sin(\mu) = 0$$

$$\iff \cos(\mu) = \mu \sin(\mu) \quad c_1 \neq 0$$

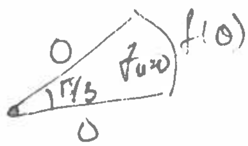
$$\iff \tan(\mu) = \frac{1}{\mu} \quad \ldots (1)$$



We see that there are multiple values μ_n that satisfy (1), & $\therefore$ multiple solutions

$$F_n(x) = \cos(\mu_n x), \quad n \in \mathbb{N}$$

4. Laplace's equation on a wedge:



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 < r < 1, \quad 0 \le \theta \le \frac{\pi}{3}, \tag{1}$$

$$u(r,0) = u\left(r, \frac{\pi}{3}\right) = 0, \quad u(1,\theta) = f(\theta), \qquad 0 < r < 1, \quad 0 \le \theta \le \frac{\pi}{3}. \tag{2}$$

By assuming $u(r,\theta) = F(r)G(\theta)$, we reduce the PDE problem above to a pair of ODE problems. Solving, we find that the eigenvalues and eigenfunctions of the θ problem are

$$\lambda_n = (3n)^2, \qquad G_n(\theta) = \sin(3n\theta), \quad n \in \mathbb{N} \tag{3}$$

The corresponding problem in r is

$$r^2 F'' + rF' - \lambda F = 0. \tag{4}$$

6 (a) Solve for $F(r)$.

Solutions are of the form $F(r) = r^\alpha$. Plugging this into (4) we have

$$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - \lambda r^\alpha = 0 \iff$$

$$\iff \alpha(\alpha-1)r^\alpha + \alpha r^\alpha - \lambda r^\alpha = 0$$

$$\iff \alpha^2 - \alpha + \alpha - \lambda = 0 \iff \alpha^2 - \lambda = 0 \iff \alpha = \pm\sqrt{\lambda}$$

$$\iff \alpha = \pm\sqrt{\lambda} = \pm 3n, \; n \in \mathbb{N}$$

$$\therefore F_n(r) = a_n r^{3n} + b_n r^{-3n}$$

On the wedge, r takes on the value zero, so we need to set $b_n = 0$ so that $F_n(0)$ is finite. Thus

$$\boxed{F_n(r) = a_n r^{3n}}$$

3 (b) Write the formal solution for $u(r,\theta)$ and include the formula for the unknown Fourier coefficients.

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^{3n} \sin(3n\theta)$$

where $$a_n = \frac{6}{\pi}\int_0^{\pi/3} f(\theta)\sin(3n\theta)\,d\theta$$

$\boxed{6}$ 5. Consider the functions

$$\left\{ \cos\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty} \tag{5}$$

Show that these functions form an orthogonal set on the interval $0 \leq x \leq L$ with respect to the weight function $w(x) = 1$.

Consider

$$I = \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \int_0^L \frac{1}{2}\left[\cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{L}{(n+m)\pi} \sin\left(\frac{(n+m)\pi x}{L}\right) + \frac{L}{(n-m)\pi} \sin\left(\frac{(n-m)\pi x}{L}\right)\right]_0^L$$

$$= 0$$

Now consider

$$II = \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx = \int_0^L \frac{1}{2}\left(1 + \cos\left(\frac{2n\pi x}{L}\right)\right) dx$$

$$= \frac{L}{2} + \frac{1}{2}\,\frac{L}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right)\Bigg|_0^L$$

$$= \frac{L}{2}$$

$\because$ $I = 0$ and $II \neq 0$, we conclude that the given set of functions is orthogonal.

$\boxed{5}$ 6. Given

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha(\frac{n\pi}{L})^2 t}, \text{ on } 0 < x < L, \tag{6}$$

where

$$u(x,0) = f(x) = \frac{5}{3}\sin\left(\frac{6\pi x}{L}\right) + 3, \tag{7}$$

find the Fourier coefficients c_n and write out the full solution for $u(x,t)$.

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = \underbrace{\frac{5}{3}\sin\left(\frac{6\pi x}{L}\right)}_{f_1(x)} + \underbrace{3}_{f_2(x)}$$

We can write

$$f_1(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where } a_n = \begin{cases} \frac{5}{3} & \text{if } n=6 \\ 0 & \text{else} \end{cases}$$

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where } b_n = \frac{2}{L}\int_0^L 3\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{6}{L}\left(\frac{-L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$= \frac{-6}{n\pi}\left[(-1)^n - 1\right]$$

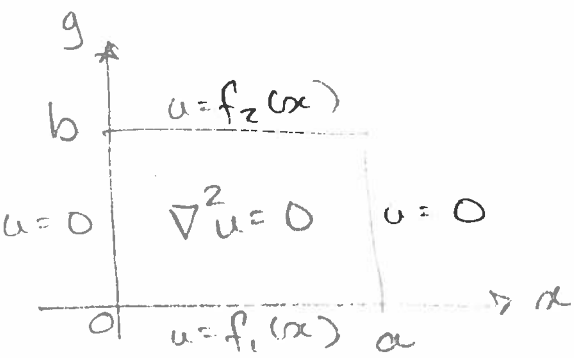
$$\therefore u(x,t) = \frac{5}{3}\sin\left(\frac{6\pi x}{L}\right)e^{-\alpha\left(\frac{6\pi}{L}\right)^2 t} - \sum_{n=1}^{\infty}\frac{6}{n\pi}\left((-1)^n - 1\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

$\boxed{5}$ 7. (*Note: This problem is a BONUS on the GROUP test. If you have extra time, you can get started on it here.*)

Consider the following Dirichlet Laplace problem on a rectangular domain:

$$\nabla^2 u = 0, \qquad 0 < x < a, \quad 0 < y < b, \tag{8}$$
$$u(0,y) = u(a,y) = 0, \qquad 0 < y < b, \tag{9}$$
$$u(x,0) = f_1(x), \quad u(x,b) = f_2(y), \qquad 0 < x < a. \tag{10}$$

Find the formal solution for $u(x,y)$ (don't solve for the Fourier coefficients).



Assume

$u(x,y) = F(x)G(y)$ .. (10a)

and plug into (8). We obtain:

$$F''G + G''F = 0 \iff \underbrace{\frac{F''}{F}}_{x\text{-dep}} = \underbrace{-\frac{G''}{G}}_{y\text{-dep}} = \underbrace{-\lambda}_{\text{constant}} \quad \ldots \quad (10b)$$

Plugging (10a) into the homogeneous BCs (9):

$$\begin{cases} F(0)G(y) = 0 \\ F(a)G(y) = 0 \end{cases} \iff \begin{cases} F(0) = 0 \\ F(a) = 0 \end{cases}$$ (for nontrivial solutions) (10c)

(10b) with (10c) gives us

Ⓐ $\begin{cases} F'' + \lambda F = 0 \\ F(0) = F(a) = 0 \end{cases}$ Ⓑ $G'' - \lambda G = 0$

We know that nontrivial solutions to Ⓐ only occur for $\lambda = \mu^2 > 0$.

(Question 7 continued.)

$$\therefore F(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

Apply the BCs:

$$F(0) = 0 \iff c_1 = 0$$

$$F(a) = 0 \iff \sin(\mu a) = 0 \iff \mu a = n\pi \iff \mu = \frac{n\pi}{a}, \; n \in \mathbb{N}$$

$$\therefore F_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

Now solve (B):

$$G_n'' - \lambda_n G_n = 0 \iff G_n'' - \left(\frac{n\pi}{a}\right)^2 G_n = 0$$

$$\iff G_n = a_n \sinh\left(\frac{n\pi y}{a}\right) + b_n \sinh\left(\frac{n\pi(y-b)}{a}\right)$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} \left[a_n \sinh\left(\frac{n\pi y}{a}\right) + b_n \sinh\left(\frac{n\pi(y-b)}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right) \quad \ldots (10d)$$

Apply the nonhomogeneous BCs:

i) $$u(x,0) = f_1(x) \iff \sum_{n=1}^{\infty} b_n \sinh\left(-\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$\therefore b_n = \frac{-1}{\sinh\left(\frac{n\pi b}{a}\right)} \frac{2}{a} \int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right) dx \quad \ldots (10e)$$

ii) $u(x,b) = f_2(x) \Leftrightarrow \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) = f_2(x)$

$$\therefore a_n = \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)} \frac{2}{a} \int_0^a f_2(x) \sin\left(\frac{n\pi x}{a}\right) dx \quad \ldots (10f)$$

Together, (10d), (10e), and (10f) form the formal solution for $u(x,y)$.