M#1 VI

Sol'n

1. $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} - u = e^{-3y}$

let:

$$\begin{cases} w = 2x - y \\ z = y \end{cases}$$

then, if $u(x,y) = v(w,z)$ we have

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 2\frac{\partial v}{\partial z}$$

and the PDE becomes

$$2\frac{\partial v}{\partial z} - v = e^{-3z} \iff \frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{1}{2}e^{-3z}$$

$$\iff \frac{\partial}{\partial z}\left(e^{-\frac{1}{2}z}v\right) = \frac{1}{2}e^{(-3-\frac{1}{2})z}$$

$$\iff e^{-\frac{1}{2}z}v = \frac{1}{2}\left(-\frac{2}{7}\right)e^{(-3-\frac{1}{2})z} + C(w)$$

$$\iff v = -\frac{1}{7}e^{-3z} + C(w)e^{\frac{1}{2}z}$$

$$\iff \boxed{u = -\frac{1}{7}e^{-3y} + C(2x-y)e^{\frac{1}{2}y}}$$

2. $5\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} - e^{x}u = \sin(y)$

let:

$$\begin{cases} w = 2x - 5y \\ z = y \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2}(w + 5z) \\ y = z \end{cases}$$

then, if $u(x,y) = v(w,z)$ we have

$$5\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 2\frac{\partial v}{\partial z}$$

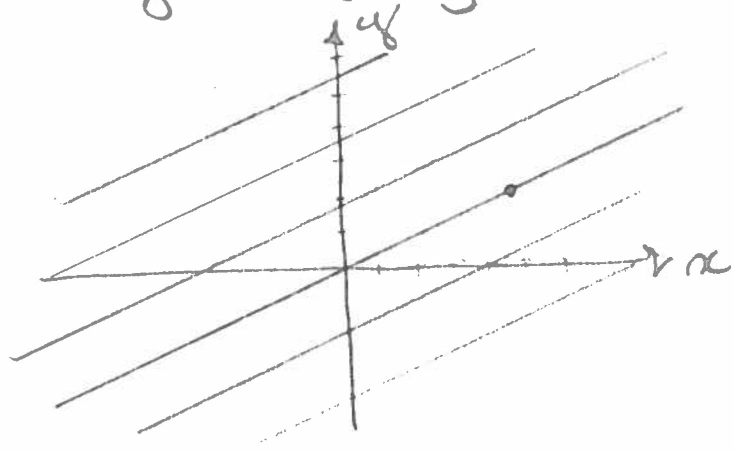
and the PDE becomes

$$\boxed{2\frac{\partial v}{\partial z} - e^{\frac{1}{2}(w+5z)}v = \sin(z).} \quad \ldots (1)$$

This is effectively an ODE.

The characteristic lines along which the ODE (1) is solved are given by

$$C = 2x - 5y \Leftrightarrow y = \frac{2}{5}x + \tilde{C}$$



3. Let

$$\begin{cases} w = bx - ay \\ z = y \end{cases}$$

and $u(x,y) = v(w,z)$. Then

$$au_x + bu_y = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y}$$

$$= a\left(\frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)$$

$$+ b\left(\frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)$$

$$= a\left(v_w b + v_z \cdot 0\right) + b\left(v_w(-a) + v_z \cdot 1\right)$$

$$= (ab - ab)v_w + bv_z$$

$$= bv_z$$

4. $u(x,t) = F(x)G(t)$

So the PDE becomes

$$F(x)G'(t) = \mathcal{D}F''(x)G(t) - v F'(x)G(t) + \beta \sin(x) F(x) G(t)$$

$$\Leftrightarrow \underbrace{\frac{G'(t)}{\mathcal{D}G(t)}}_{\text{depends on } t} = \underbrace{\frac{F''(x)}{F(x)} - v\frac{F'(x)}{F(x)} + \beta \sin(x)}_{\text{depends on } x} \quad \ldots (1)$$

$\because$ x + t are lin. indep., (1) can only be true if each side is equal to a constant. $\therefore$ we have

$$\frac{G'(t)}{\mathcal{D}G(t)} = \frac{F''(x)}{F(x)} - v\frac{F'(x)}{F(x)} + \beta \sin(x) = -\lambda$$

which gives us

$$\begin{cases} F''(x) - vF'(x) + (\beta \sin(x) + \lambda)F(x) = 0 \\ G'(t) + \lambda \mathcal{D} G(t) = 0 \end{cases}$$

Now we separate the BCs & ICs

$$\begin{cases} \dfrac{\partial u}{\partial x}(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(4\pi, t) = 0 \end{cases} \Leftrightarrow \begin{cases} F'(0)G(t) = 0 \\ F'(4\pi)G(t) = 0 \end{cases}$$

For nontrivial solns we require $F'(0) = F'(4\pi) = 0$

no harmonics.

$\therefore$ the ODE problems are:

$$\text{(A)} \begin{cases} F''(x) - v F'(x) + (\beta \sin(x) + \lambda) F(x) = 0, \\ F'(0) = F'(4\pi) = 0, \end{cases}$$

and

$$\text{(B)} \quad G'(t) + \lambda D G(t) = 0.$$

5. a) $r^2 + r + \lambda = 0 \iff r = \dfrac{-1 \pm \sqrt{1-4\lambda}}{2}$

case 1: $1 - 4\lambda > 0 \iff \lambda < \frac{1}{4}$; write $1 - 4\lambda = \rho^2$
then $r = -\frac{1}{2} \pm \frac{\rho}{2}$, both real and

$$F(x) = e^{-\frac{1}{2}x} \left(c_1 \cosh\left(\frac{\rho}{2}\right) + c_2 \sinh\left(\frac{\rho}{2}\right) \right)$$

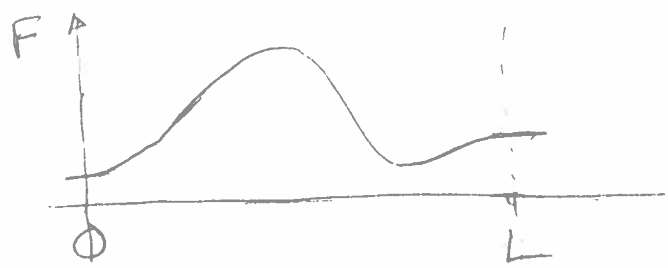
case 2: $1 - 4\lambda = 0 \iff \lambda = \frac{1}{4}$
then $r = -\frac{1}{2}$, double real root and

$$F(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}$$

case 3: $1-4\lambda < 0 \Leftrightarrow \lambda > \frac{1}{4}$; write $1-4\lambda = -\rho^2$

then $r = -\frac{1}{2} \pm i\frac{\rho}{2}$, and

$$F(x) = e^{-\frac{1}{2}x}\left(c_1 \cos\left(\frac{\rho}{2}\right) + c_2 \sin\left(\frac{\rho}{2}\right)\right)$$

b) 

We know that the slope at 0 & L has to be zero from (2). In case 1, the solutions are exponential curves which do not have a zero derivative on a finite domain. In case 2, the solutions are decaying exponentials times polynomials, which can only have a zero derivative at one point in the domain. Thus, in cases 1 & 2, we cannot satisfy the BCs (2).

In case 3, the solutions are sinusoidal. Those functions have zero slope at multiple values of the argument. In this case, therefore, nontrivial solutions are possible.

6. a) $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi\alpha t}{2}\right)$

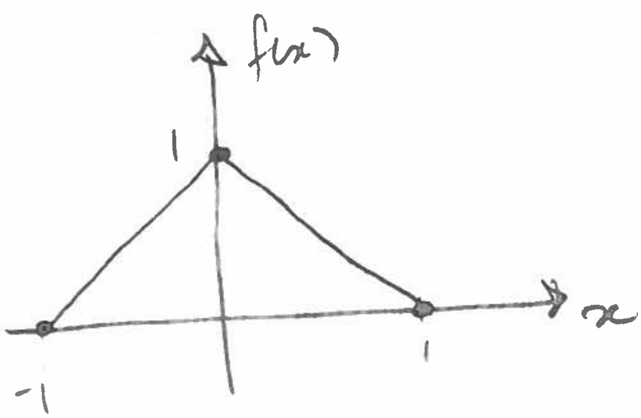
b) $\frac{\partial u}{\partial t}(x,0) = f(x) \Leftrightarrow f(x) = \sum_{n=1}^{\infty} \frac{n\pi\alpha}{2} B_n \sin\left(\frac{n\pi x}{2}\right)$

This is a Fourier sine series for $f(x)$.

7. $f(x) = 1 - |x|, \qquad -1 < x < 1$

$$= \begin{cases} 1-(-x) & -1 < x \leq 0 \\ 1-x & 0 < x < 1 \end{cases}$$

$$= \begin{cases} 1+x & -1 < x \leq 0 \\ 1-x & 0 < x < 1 \end{cases}$$



f(x) is symmetric about the origin and is defined on a symmetric interval.
So we extend f(x) by repeating it + write

$$f_r(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

where $a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx$

9.

$$a_n = \int_{-1}^{0} (1+x)\cos(n\pi x)\,dx + \int_{0}^{1} (1-x)\cos(n\pi x)\,dx$$

$$= \int_{-1}^{0} \cos(n\pi x)\,dx + \int_{-1}^{0} \underbrace{\overset{\text{odd}}{x}\,\overset{\text{even}}{\cos(n\pi x)}}_{\text{odd}}\,dx$$

$$+ \int_{0}^{1} \cos(n\pi x)\,dx - \int_{0}^{1} x\cos(n\pi x)\,dx$$

$$= \overbrace{\int_{-1}^{1} \cos(n\pi x)\,dx}^{0} - 2\int_{0}^{1} x\cos(n\pi x)\,dx$$

$$= -2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= -2\left[\frac{1}{n\pi}\overbrace{\sin(n\pi)}^{0} + \frac{1}{(n\pi)^2}\overbrace{\cos(n\pi)}^{(-1)^n}\right.$$

$$\left. - 0 - \frac{1}{(n\pi)^2}\overbrace{\cos(0)}^{1}\right]$$

$$= \frac{-2}{(n\pi)^2}\left[\cos(n\pi) - 1\right] \quad \text{OR} \quad \frac{2}{(n\pi)^2}\left[1 - (-1)^n\right]$$

$\because$ we can't use $n=0$ in the formula for a_n, we compute a_0 separately:

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}(1+x)\,dx + \int_{0}^{1}(1-x)\,dx$$

$$= \int_{-1}^{1} 1\,dx - 2\int_{0}^{1} x\,dx = x\Big|_{-1}^{1} - x^2\Big|_{0}^{1}$$

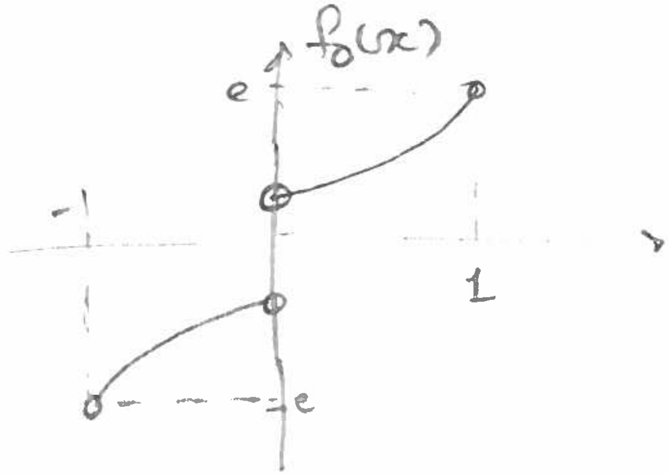
$$= 2 - 1 = 1$$

$$\therefore f_r(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\left[1-(-1)^n\right]\cos(n\pi x)$$

OR

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{(n\pi)^2}\left[\cos(n\pi)-1\right]\cos(n\pi x)$$

8. $f(x) = e^x, \quad 0 < x < 1$



odd fn, $\therefore a_n = 0 \; \forall n$

$$b_n = \frac{2}{1} \int_0^1 f_o(x) \sin\left(\frac{n\pi x}{1}\right) dx$$

$$= 2 \int_0^1 e^x \sin(n\pi x)\, dx$$

$$= 2 \left[\frac{e^x}{(n\pi)^2+1} \left(\sin(n\pi x) - n\pi \cos(n\pi x) \right) \right]_0^1$$

$$= \frac{2}{(n\pi)^2+1} \left[e\left(\overset{0}{\cancel{\sin(n\pi)}} - n\pi\cos(n\pi) \right) - \overset{1}{\cancel{e^0}} \left(\overset{0}{\cancel{\sin(0)}} - n\pi \overset{1}{\cancel{\cos(0)}} \right) \right]$$

$$= \frac{2}{(n\pi)^2+1} \left[-en\pi\cos(n\pi) + n\pi \right] = \frac{2n\pi}{(n\pi)^2+1} \left[1 - e\cos(n\pi) \right]$$

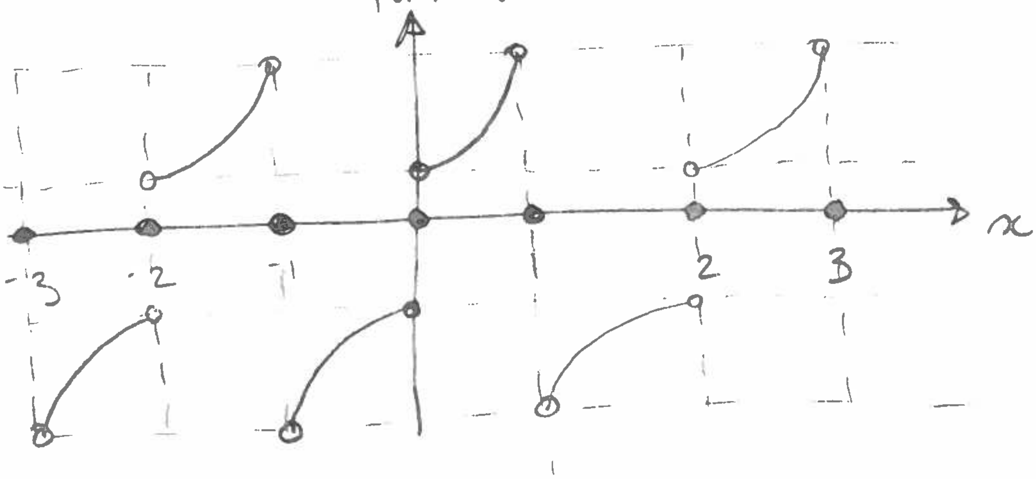
OR $\dfrac{2n\pi}{(n\pi)^2+1} \left[1 - e(-1)^n \right]$

$$\therefore \; f_o(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{(n\pi)^2+1}\left[1 - e\cos(n\pi)\right]\sin(n\pi x)$$

OR

$$\sum_{n=1}^{\infty} \frac{2n\pi}{(n\pi)^2+1}\left[1 - e(-1)^n\right]\sin(n\pi x)$$

$f(x)$ extended (w/o the filled circles)



$f_o(x)$ tends to $f(x)$ as shown + the filled circles.