MIDTERM #1 SAMPLE TEST PROBLEMS

COURSE: Differential Equations II (PDEs)
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Disclaimer: This set of sample problems is too long for a midterm test. The actual test would be a subset of problems of the general type that appear here. These problems are provided as a study resource, not as a summary of the course taught so far. Problems on the midterm can come from **any material** in the lectures, assignments and pre-reading assignments. These problems however, should give you an idea of the way PDE questions can be asked so as to be doable in the 45-minute timeframe of the midterm.

Note: Figures and useful integrals appear on the last page!

5 1. Use the method of characteristics to find the general solution of

$$u_x + 2u_y - u = e^{-3y}. \tag{1}$$

5 2. Use the method of characteristics to transform the PDE below into an ODE. Sketch the characteristic lines. *Note: Do not solve the ODE!!*

$$5u_x + 2u_y - e^x u = \sin(y) \tag{2}$$

4 3. Consider the expression

$$au_x + bu_y \tag{3}$$

where a and b are constants. Prove that (3) can be transformed to the simpler expression, bv_z, using an appropriate transformation of variables. *Note: For this problem, show every step!*

5 4. Use the technique of "separation of variables" to transform the PDE problem below into a pair of ODE problems with boundary and initial conditions if these are separable too. *(Note: Simply state the ODE problems; do not solve them!)*

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} + \beta \sin(x)u \text{ on } 0 < x < 4\pi, \tag{4a}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(4\pi, t) = 0, \tag{4b}$$

$$u(x,0) = f(x). \tag{4c}$$

5. Consider the BVP

$$F''(x) + F'(x) + \lambda F(x) = 0, \qquad 0 < x < L, \tag{5a}$$

$$F'(0) = F'(L) = 0. \tag{5b}$$

4 (a) First consider just the ODE (5a). Use the roots of the characteristic equation to find the three different types of general solution that (5a) admits, depending on the value of λ. *(Note: Use $1 - 4\lambda = \rho^2$ or $1 - 4\lambda = -\rho^2$, as appropriate, to simplify your answers.)*

2 (b) Now considering the boundary values (5b), which of the three solution types that you found in a will yield nontrivial solutions? Why? Convince me that you're not guessing!

6. Consider the PDE problem

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2, \quad t > 0 \tag{6a}$$

$$u(0,t) = u(2,t) = 0, \tag{6b}$$

$$u(x,0) = 0, \tag{6c}$$

$$\frac{\partial u}{\partial t}(x,0) = f(x). \tag{6d}$$

By applying the assumption $u(x,t) = F(x)G(t)$, and using (6a)-(6c), we find that

$$F_n(x) = c_n \sin\left(\frac{n\pi x}{2}\right), \qquad G_n(t) = d_n \sin\left(\frac{n\pi\alpha t}{2}\right), \qquad n = 1,\, 2,\, 3,\, \ldots \tag{7}$$

1 (a) What is the general solution for $u(x,t)$?

1 (b) Apply the initial condition (6d). Whast is the series you obtain for $f(x)$? Name it.

6 7. Find the Fourier series representation for the function

$$f(x) = 1 - |x|, \qquad -1 < x < 1. \tag{8}$$

The function is shown in Figure 1. *(Note: Use the sketch of the function to sipmlify your work!)*

8. Consider the function

$$f(x) = e^x, \qquad 0 < x < 1. \tag{9}$$

A sketch of the function is shown in Figure 2.

4 (a) Find the Fourier sine series representation for $f(x)$.

3 (b) On the interval [-3,3], sketch the original function, extended as appropriate, and the function to which your series converges. *Note: The function $f_o(x)$ is not continuous. Make sure that all points of discontinuity are clearly marked!*

Figures and Useful Information

Some integrals you may find useful:

$$\int x \sin(\rho x)\, dx = -\frac{x}{\rho}\cos(\rho x) + \frac{1}{\rho^2}\sin(\rho x) \tag{10}$$

$$\int x \cos(\rho x)\, dx = \frac{x}{\rho}\sin(\rho x) + \frac{1}{\rho^2}\cos(\rho x) \tag{11}$$

$$\int e^x \sin(\rho x)\, dx = \frac{e^x}{\rho^2+1}\left[\sin(\rho x) - \rho\cos(\rho x)\right] \quad \int e^x \cos(\rho x)\, dx = \frac{e^x}{\rho^2+1}\left[\rho\sin(\rho x) + \cos(\rho x)\right] \tag{12}$$

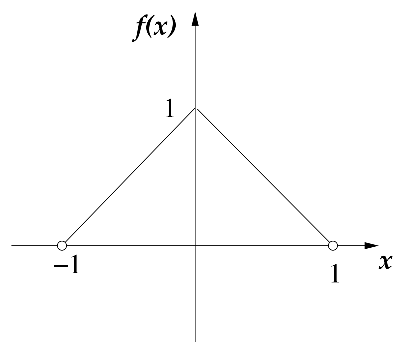

Figure 1: Plot of $f(x)$ for question 7.

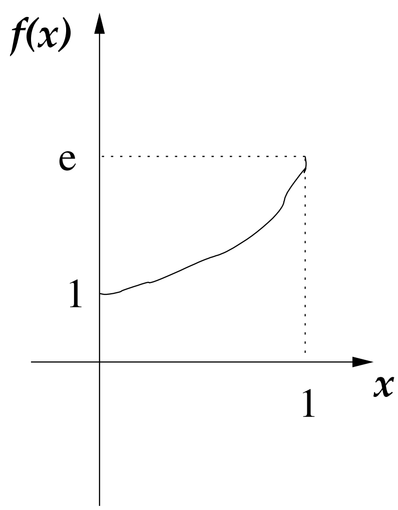

Figure 2: Plot of $f(x)$ for question 8.