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THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 319
Date: Dec 12th, 2014 Time: 6:00pm Duration: 3 hours.
This exam has 7 questions for a total of 58 points.

NAME: Solutions

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

This exam is an individual exam only. You have 3 hours to complete the exam.

Problem Number	1	2	3	4	5	6	7	Total
Points Earned								
Points Out Of	8	3	9	6	12	7	10	58

1. Consider the PDE

$$3\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} + u = x. \qquad (1)$$

2 (a) Find the equation for the characteristic lines of (1).

Slope $= \frac{b}{a} = -\frac{2}{3}$

$\therefore$ the characteristic lines are the parallel lines $y = -\frac{2}{3}x + C \iff 2x + 3y = c$

2 (b) Use a change of variables $v(w, z) = u(x, y)$ to convert the PDE problem (1) into the ODE problem

$$\frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{1}{4}(3z - w). \qquad (2)$$

Show all of your work.

let $\begin{cases} w = 2x + 3y \\ z = y \end{cases} \iff \begin{cases} x = \frac{w - 3z}{2} \\ y = z \end{cases}$

then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} = 2v_w$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} = 3v_w + v_z$$

Plugging these into (1) we obtain

$$3(2v_w) - 2(3v_w + v_z) + v = \frac{w - 3z}{2} \iff$$

$$\iff -2v_z + v = \frac{w-3z}{2} \iff \boxed{v_z - \frac{1}{2}v = \frac{3z - w}{4}}$$

4 (c) Solve (2). Give the final answer in terms of x and y.

We first look for an integrating factor

$$\mu(z) = e^{\int -\frac{1}{2} dz} = e^{-\int \frac{1}{2} dz} = e^{-\frac{1}{2}z}$$

Then (2) becomes

$$\frac{d}{dz}\left(e^{-\frac{1}{2}z} v\right) = \frac{3z - w}{4} e^{-\frac{1}{2}z} \quad (1)$$

$$(1) \Leftrightarrow e^{-\frac{1}{2}z} v = \frac{1}{4}\int \left(3z e^{-\frac{1}{2}z} - w e^{-\frac{1}{2}z}\right) dz$$

$$= \frac{1}{4}\left[-3(2) z e^{-\frac{1}{2}z} + 2\int e^{-\frac{1}{2}z} dz + 2we^{-\frac{1}{2}z}\right] + C(w)$$

$$= \frac{-1}{\cancel{4}_{2}}\left[+\cancel{6}^{3} z e^{-\frac{1}{2}z} + \cancel{4}^{2} e^{-\frac{1}{2}z} + \cancel{4}^{2} w e^{-\frac{1}{2}z}\right] + C(w)$$

$$\Leftrightarrow v(w,z) = -\frac{1}{2}(3z + 2 + 2w) + C(w)e^{\frac{1}{2}z}$$

$$\therefore u(x,y) = \frac{-1}{2}\left(3y + 2 + 2(2x + 3y)\right) + C(2x + 3y)e^{\frac{1}{2}y}$$

$$= -\frac{1}{2}(9y + 4x + 2) + C(2x + 3y)e^{\frac{1}{2}y}$$

$\boxed{6}$ 2. Consider the BVP

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + \lambda y = 0, \qquad 0 < x < 2, \tag{3a}$$

$$y(0) = 0, \quad \frac{dy}{dx}(2) = 0. \tag{3b}$$

Find the values of λ for which the given problem has nontrivial solutions. Check all cases and write down the eigenfunctions.

char. eqn:

$r^2 - 2r + \lambda = 0 \iff r = 1 \pm \sqrt{1-\lambda}$

case ①: $1-\lambda > 0 \iff 1-\lambda = \mu^2,\ \mu > 0$

$r = 1 \pm \mu$, and $y(x) = e^x(A\cosh(\mu x) + B\sinh(\mu x))$

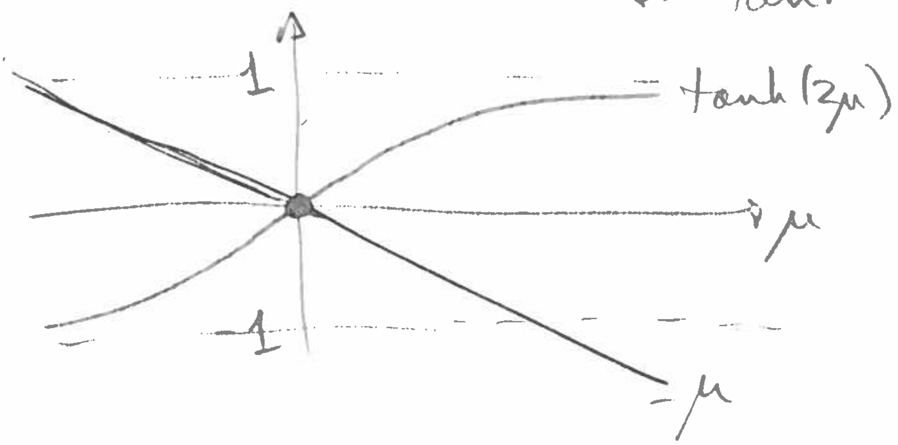
apply the BCs

$y(0) = 0 \iff 1(A\cdot 1 + B\cdot 0) = 0 \iff A = 0$

$y'(2) = 0 \iff e^x(B\sinh(\mu x))\Big|_2 + e^x B\mu\cosh(\mu x)\Big|_2 = 0$

$\iff \cancel{e^2}\left[B\sinh(2\mu) + B\mu\cosh(2\mu)\right] = 0$

$\iff \tanh(2\mu) = -\mu$



The only intersection is at $\mu = 0$. But we assumed $\mu > 0$ $\therefore$ there are no nontrivial solutions for this case.

(Additional workspace for problem # 2.)

<u>case ②</u>: $1-\lambda=0 \iff \lambda=1$

$r=1$ and $y(x)=c_1e^x+c_2xe^x$

apply the BCs:

$y(0)=0 \iff c_1=0$

$y'(2)=0 \iff c_2\,2e^2=0 \iff c_2=0$

$\therefore$ We only have trivial solutions in this case.

<u>case ③</u>: $1-\lambda<0 \iff 1-\lambda=-\mu^2,\ \mu>0$

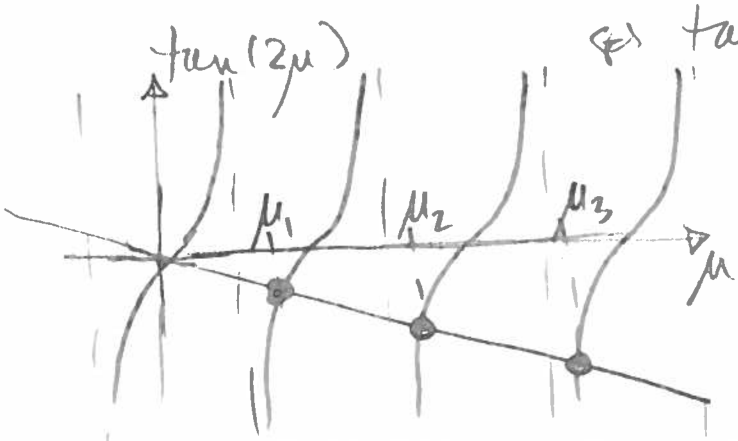
$r=1\pm i\mu$ and $y(x)=e^x\left(A\cos(\mu x)+B\sin(\mu x)\right)$

apply BCs:

$y(0)=0 \iff A=0$

$y'(2)=0 \iff \left[\cancel{e^x}\cancel{B}\sin(\mu x)+\mu\cancel{e^x}\cancel{B}\cos(\mu x)\right]\Big|_{x=2}=0$

$\iff \tan(2\mu)=-\mu$



We see that there are multiple solutions $\mu_i>0$, $\therefore$ the eigenvalu are $\lambda_n=-\mu_n^2$. The eigenfns are

$y_n(x)=B_ne^x\sin(\mu_n x)$

3. Consider

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0, \tag{4a}$$

$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \qquad t > 0 \tag{4b}$$

$$u(x,0) = f(x), \qquad 0 < x < L. \tag{4c}$$

where

$$f(x) = \sin(4x) + 3\sin(6x) - \sin(10x). \tag{5}$$

$\boxed{4}$ (a) Apply separation of variables (use $F(x)$ and $G(t)$) to turn this PDE problem into a pair of ODE problems.

$u = F(x)G(t)$

$\therefore (4a) \Leftrightarrow \frac{FG'}{\beta FG} = \frac{\beta F''G}{\beta FG} \Leftrightarrow \frac{1}{\beta}\frac{G'}{G} = \frac{F''}{F} = -\lambda$

$\therefore \begin{cases} F'' + \lambda F = 0 \\ F(0) = F'(L) = 0 \end{cases} \qquad \begin{cases} G' + \lambda \beta G = 0 \\ \end{cases}$

(this is the BVP)

$\boxed{5}$ (b) Given that the eigenvalues and eigenfunctions of the BVP in part (a) are

$$\lambda_n = n^2, \quad n \in \mathbb{N}, \qquad F_n(x) = \sin(nx), \tag{6}$$

find the solution to the system (4). (*Note: This means that the hard part of "step 3" has been done for you!*)

$u(x,t) = \sum_{n=1}^{\infty} a_n F_n(x) G_n(t)$

(Additional workspace for problem # 3.)

We need $G_n(t)$:

$$G' + n^2 \beta G = 0 \Leftrightarrow \frac{dG}{dt} = -n^2 \beta G \Leftrightarrow$$

$$\Leftrightarrow \int \frac{dG}{G} = \int -n^2 \beta \, dt \Leftrightarrow \ln|G| = -n^2 \beta t + K$$

$$\Leftrightarrow |G_n| = A e^{-n^2 \beta t}$$

we take $A \geq 0$

$$G_n = A e^{-n^2 \beta t}$$

$$\boxed{\therefore u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx) e^{-n^2 \beta t}}$$

Now apply the IC

$$u(x,0) = f(x) \Leftrightarrow f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$$\boxed{\therefore a_n = \frac{2}{L} \int_0^L f(x) \sin(nx) \, dx}$$

These two results give the solution.

4. Consider the function

$$f(x) = \begin{cases} -1 & 0 \leq x < 1, \\ 2 & 1. \end{cases} \tag{7}$$

A plot of the function appears in Figure 1.

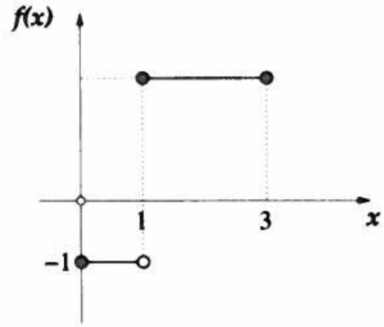


Figure 1: Plot of $f(x)$ for question 4.

$\boxed{2}$ (a) Let $F(x)$ be the function to which the fourier sine representation of $f(x)$ converges. Sketch $F(x)$ on the axes below.

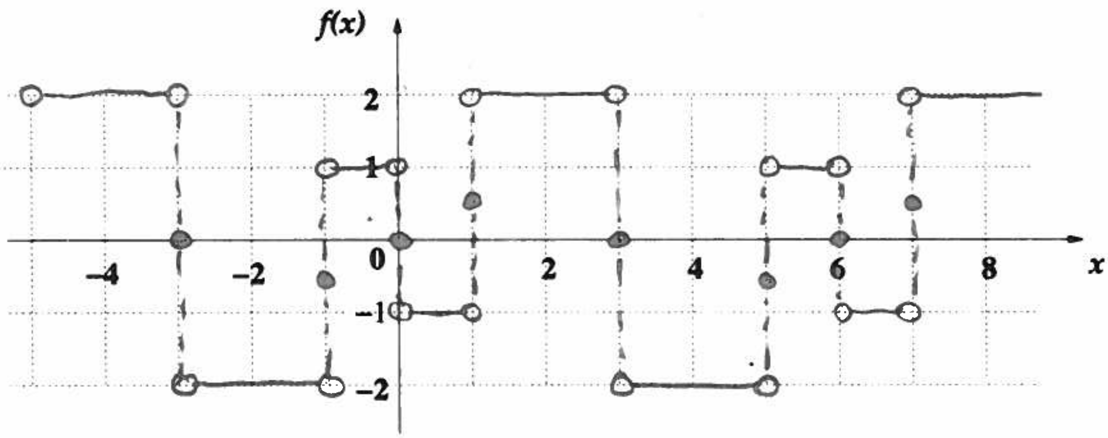


Figure 2: Axes for question 4(a).

4 (b) Let $G(x)$ be the function to which the fourier cosine representation of $f(x)$ converges. Compute $G(x)$.

$$G(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right)$$

where

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\left[\int_0^1 -1\,dx + \int_1^3 2\,dx\right] = \frac{2}{3}\left[-x\Big|_0^1 + 2x\Big|_1^3\right]$$

$$= \frac{2}{3}\left[-1 + 2(3-1)\right] = \frac{2}{3}\left[-1+4\right] = 2$$

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\int_0^1 -\cos\left(\frac{n\pi x}{3}\right)dx + \int_1^3 2\cos\left(\frac{n\pi x}{3}\right)dx\right]$$

$$= \frac{2}{3}\frac{(-3)}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Big|_0^1 + \frac{4}{3}\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Big|_1^3$$

$$= -\frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right) + \frac{4}{n\pi}\left[\cancel{\sin(n\pi)}^{\,0} - \sin\left(\frac{n\pi}{3}\right)\right]$$

$$= \frac{-6}{n\pi}\sin\left(\frac{n\pi}{3}\right)$$

$$\therefore \boxed{G(x) = 1 - \sum_{n=1}^{\infty}\frac{6}{n\pi}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi x}{3}\right)}$$

4 ~~(b) Let $G(x)$ be the function to which the fourier cosine representation of $f(x)$ converges. Compute $G(x)$.~~

12 5. Find a solution for the Neumann BVP on the disk

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 < r < a, \quad -\pi \le \theta \le \pi, \tag{8a}$$

$$\frac{\partial u}{\partial r}(a,\theta) = f(\theta), \qquad -\pi \le \theta \le \pi. \tag{8b}$$

In solving the BVP in θ, you can assume that the eigenvalue is positive or zero. *Hint: Recall that $u(r,\theta)$ should be continuous at $\theta = -\pi$ and $\theta = \pi$, and that $u(r,\theta)$ must be finite at the origin.*

Step 1 Assume $u(r,\theta) = R(r)T(\theta)$ & plug into (8a)

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0 \iff \frac{r^2R''}{R} + \frac{rR'}{R} = \frac{-T''}{T} = \lambda$$

$\therefore$ we arrive at two ODE problems

Step 2

$$\text{(A)} \begin{cases} T'' + \lambda T = 0 \\ T(-\pi) = T(\pi) \\ T'(-\pi) = T'(\pi) \end{cases} \qquad \text{(B)} \quad r^2R'' + rR' - \lambda R = 0$$

$\because T(\theta)$ must be periodic

Given: $\lambda \ge 0$

Step 3

Solving (A) we have

Case 1: $\lambda > 0$, let $\lambda = \mu^2$, then we have

$p^2 + \mu^2 = 0 \iff p = \pm i\mu$ and $T(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$

apply the BCs:

We require $T(\theta)$ to be 2π-periodic. In order for this to be true, we require $\mu = n \in \mathbb{N}$.

(Additional workspace for problem # 5.)

case ②: $\lambda = 0$

$p^2 + \lambda = 0 \Leftrightarrow p = 0 \quad \therefore \quad T(\theta) = A_0 + B_0\theta$

apply the BCs:

For $T(\theta)$ to be 2π-periodic, we require $B_0 = 0$.

So we have

eigenvalues: $\lambda_0 = 0$; $\lambda_n = n^2, n \in \mathbb{N}$

eigenfunctions: $T_0(\theta) = A_0$; $T_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$

Now we solve Ⓑ.

case(2): $\lambda = 0$

$r^2R'' + rR' = 0$; let $Q = R'$ then we have

$r^2Q' + rQ = 0 \Leftrightarrow \begin{cases} r = 0 \text{ or} \\ \frac{Q'}{Q} = \frac{-1}{r} \end{cases} \Leftrightarrow \int \frac{dQ}{Q} = -\int \frac{1}{r}\,dr$

$\Leftrightarrow \ln|Q| = -\ln|r| + C$

$\Leftrightarrow \ln|Q| = \ln\left(\frac{1}{r}\right) + C \quad , \because r > 0$

choosing appropriately, we have $Q = \frac{D_0}{r}$, D_0 cst.

$\therefore R' = \frac{D_0}{r} \Leftrightarrow R = D_0\ln(r) + C_0$

for bounded solutions as $r \to 0$ we require $D_0 = 0$

(Additional workspace for problem # 5,)

Case ①: $\lambda = n^2$, $n \in \mathbb{N}$

$r^2 R'' + rR' - n^2 R = 0$ This is a Cauchy-Euler equation with solutions of the form r^p so the characteristic equation is

$$p(p-1) + p - n^2 = 0 \text{ or } p^2 - n^2 = 0 \text{ or } p = \pm n$$

$\therefore$ solutions are

$$R(r) = C_n r^n + D_n r^{-n}$$

For bounded solutions as $r \to 0$ we require $D_n = 0$.

Step ④

$$u_n(r,\theta) = R_n(r)T_n(\theta) = C_n r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

$n \in \mathbb{N}_0$ (Note: $B_0 = 0$)

$$\therefore \boxed{u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)}$$

Step ⑤

apply the BC (8b):

$\frac{\partial u}{\partial r}(a,\theta) = f(\theta)$ or $\boxed{f(\theta) = \sum_{n=1}^{\infty} \frac{n}{a} \left(\frac{a}{a}\right)^{n-1} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)}$

This is the FORMAL solution.

6. Consider

$$(xy')' + \frac{\mu^2}{x} y = 0, \qquad 1 < x < e, \quad \mu > 0, \quad \mu \in \mathbb{R} \tag{9a}$$

$$y'(1) = 0, \quad y(e) = 0. \tag{9b}$$

$\boxed{2}$ (a) Verify that the problem (9) is a Regular Sturm-Liouville BVP and give the coefficients.

This is an S-L problem with
$p(x) = x$, $q(x) = 0$, $r(x) = \frac{1}{x}$, $a_1 = 0, b_1 = 1, a_2 = 1, b_2 = 0$
and $p(x)$ + $r(x)$ are continuous,
and $p(x)$ + $r(x)$ are both greater than zero
on $[1, e]$.

$\boxed{5}$ (b) Compute the solution.

$(9a) \Leftrightarrow xy'' + y' + \frac{\mu^2}{x} y = 0 \Leftrightarrow x^2 y'' + xy' + \mu^2 y = 0$

Let $y = x^\alpha$, then we have
$x^2 \alpha(\alpha-1) x^{\alpha-2} + x \alpha x^{\alpha-1} + \mu^2 x^\alpha = 0 \Leftrightarrow$ ⁄
⁄ $\Leftrightarrow x^\alpha (\alpha(\alpha-1) + \alpha + \mu^2) = 0 \Leftrightarrow \alpha^2 - \alpha + \alpha - \mu^2 = 0$
$\Leftrightarrow \alpha^2 + \mu^2 = 0 \Leftrightarrow \mu = \pm i\alpha$

$\therefore y = A_0 \cos(\alpha \ln(x)) + B_0 \sin(\alpha \ln(x))$
$y' = -A_0 \frac{\alpha}{x} \sin(\alpha \ln(x)) + B_0 \frac{\alpha}{x} \cos(\alpha \ln(x))$

Apply (9b):
$y'(1) = 0 \Leftrightarrow -A_0 \alpha \overset{\to 0}{(0)} + B_0 \alpha (1) = 0 \Leftrightarrow B_0 = 0$

$$y(e)=0 \Leftrightarrow A_0 \cos(\alpha)=0 \Leftrightarrow \alpha = \frac{(2n-1)\pi}{2}, \quad n \in \mathbb{N}$$

$$\therefore \quad y_n(x) = c_n \cos\left(\frac{(2n-1)\pi}{2} \ln(x)\right)$$

and $$\boxed{y(x) = \sum_{n=1}^{\infty} c_n \cos\left(\left(\frac{2n-1}{2}\right)\pi \ln(x)\right)}$$

$\boxed{10}$ 7. Find the eigenfunction expansion for the solution to the following non-homogeneous BVP:

$$y'' + 2y = \pi x, \qquad 0 < x < \pi, \tag{10a}$$
$$y(0) = y(\pi) = 0. \tag{10b}$$

(Additional workspace for problem # 7.)

[10] 7. Find the eigenfunction expansion for the solution to the following non-homogeneous BVP:

$$y'' + 2y = \pi x, \qquad 0 < x < \pi, \tag{10a}$$
$$y(0) = y(\pi) = 0. \tag{10b}$$

To obtain an eigenfunction expansion for the solution ϕ to the nonhomogeneous regular Sturm-Liouville BVP

$$L[y] + \mu r y = f, \qquad B[y] = 0$$

1) Find an orthogonal system of eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ and corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ for

$$L[y] + \lambda r y = 0, \qquad B[y] = 0.$$

For our problem, we have

$$\mu = 2, \quad r(x) = 1, \quad f(x) = \pi x.$$

Consider the problem

$$y'' + \lambda y = 0, \qquad y(0) = y(\pi) = 0.$$

Set

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$$y'' + \lambda y = 0$$
$$r^2 e^{rx} + \lambda e^{rx} = 0$$
$$r^2 + \lambda = 0$$
$$r = \pm\sqrt{-\lambda}.$$

Case 1: $\lambda = 0$

$y'' = 0$

$y' = A$

$y = Ax + B$

$$y(0) = A(0) + B = 0$$
$$B = 0$$
$$y(\pi) = A \cdot \pi = 0$$
$$A = 0$$

Trivial Solution

(Additional workspace for problem # 7.)

Case 2: $\lambda<0$, set $\lambda=-s^2$, so $r=\pm s$

$y(x)=c_1\cosh(sx)+c_2\sinh(sx)$

$y(0)=c_1\cdot 1+c_2\cdot 0=0$

$\Rightarrow c_1=0$.

$y(x)=c_2\sinh(sx)$

$y(\pi)=c_2\underbrace{\sinh(s\pi)}_{\neq 0}=0$ Trivial Solution

$c_2=0$

Case 3: $\lambda>0$, set $\lambda=s^2$, so $r=\pm is$

$y(x)=c_3\cos(sx)+c_4\sin(sx)$

$y(0)=c_3\cdot 1+c_4\cdot 0=0$

$\Rightarrow c_3=0$

$y(x)=c_4\sin(sx)$

$y(\pi)=c_4\sin(s\pi)=0$.

$\hookrightarrow \sin(s\pi)=0$

$s\pi=n\pi$ for $n=1,2,\ldots$

$s=n$

Eigenvalues: $\lambda_n=n^2$ for $n=1,2,\ldots$

Eigenfunctions: $\phi_n(x)=\sin nx$

2) Compute the eigenfunction expansion for f/r; that is determine the coefficients γ_n so that

$$f/r=\sum_{n=1}^{\infty}\gamma_n\phi_n$$

We have $r(x)=1$, $\phi_n(x)=\sin nx$, $f(x)=\pi x$

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n \phi_n$$

$$\frac{\pi x}{1} = \sum_{n=1}^{\infty} \gamma_n \sin nx.$$

Compute
$$\gamma_n = \frac{\int_0^{\pi} \frac{f(x)}{r(x)} \phi_n(x) r(x)\, dx}{\int_0^{\pi} \phi_n^2(x) r(x)\, dx} = \frac{\int_0^{\pi} \pi x \sin nx\, dx}{\int_0^{\pi} \sin^2 nx\, dx}$$

We are told

$$\int_0^{\pi} \sin^2(nx)\,dx = \frac{\pi}{2} - \frac{\sin 2n\pi}{4n} = \boxed{\frac{\pi}{2}}$$

and

$$\int x\sin(nx)\,dx = \frac{\sin(nx) - nx\cos(nx)}{n^2}$$

so

$$\pi\int_0^{\pi} x\sin(nx)\,dx = \pi\left[\frac{\cancelto{0}{\sin n\pi} - n\pi\cos(n\pi)}{n^2} - 0\right]$$

$$= \boxed{\frac{-\pi^2(-1)^n}{n}}$$

$\therefore$ $$\gamma_n = \frac{-\pi^2(-1)^n/n}{\pi/2} = \boxed{\frac{-2\pi(-1)^n}{n}}$$

Since $\mu = 2 \neq n^2 = \lambda_n$ for $n=1,2,\ldots$, then the eigenfunction expansion for the solution ϕ is

$$\phi(x) = \sum_{n=1}^{\infty} \frac{\gamma_n}{\mu-\lambda_n}\phi_n$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{-2\pi(-1)^n}{n(2-n^2)} \sin nx}$$