A1 Sol'ns

1. a) $4y'' + y' = 0$

char eqn: $4r^2 + r = 0 \Leftrightarrow r(4r+1) = 0$

$\Leftrightarrow r = 0$ or $r = -\frac{1}{4}$

$\therefore$ the general sol'n is

$$y = C_1 + C_2 e^{-\frac{1}{4}t}$$

b) $y'' - y' - 6y = 0$

char eqn: $r^2 - r - 6 = 0 \Leftrightarrow (r-3)(r+2) = 0$

$\Leftrightarrow r = 3$ or $r = -2$

$\therefore$ gen'l sol'n is

$$y = C_1 e^{3t} + C_2 e^{-2t}$$

c) $12y'' - 5y' - 2y = 0$

char eqn: $12r^2 - 5r - 2 = 0 \Leftrightarrow r = \frac{5 \pm \sqrt{25 + 96}}{24}$

$\Leftrightarrow r = \frac{5 \pm \sqrt{121}}{24}$

$= \frac{5 \pm 11}{24}$

$= \frac{16}{24}$ or $\frac{-6}{24}$

$\therefore r = \frac{2}{3}$ or $r = -\frac{1}{4}$

and the gen'l sol'n is

$$y = c_1 e^{\frac{2}{3}t} + c_2 e^{-\frac{1}{4}t}$$

2. $y'' + y' + 2y = 0, \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$

char eqn:

$$r^2 + r + 2 = 0 \Leftrightarrow r = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$$

$$\therefore y(t) = e^{-\frac{1}{2}t}\left(c_1 \cos\left(\frac{\sqrt{7}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}t\right)\right)$$

apply the ICs:

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ -\frac{1}{2}c_1 + \frac{\sqrt{7}}{2}c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ c_2 = \frac{2}{\sqrt{7}} \frac{1}{2} = \frac{1}{\sqrt{7}} \end{cases}$$

$$\therefore\ y(t) = e^{-\frac{1}{2}t}\left(\cos\left(\frac{\sqrt{7}}{2}t\right) + \frac{1}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$$

3. $y'' + \lambda y = 0 \quad \begin{cases} y(0) = 0 \\ y(\frac{\pi}{2}) = 0 \end{cases}$

char eqn:

$$r^2 + \lambda = 0 \iff r^2 = -\lambda \iff r = \pm\sqrt{-\lambda}$$

Cases

i) $\lambda < 0$ or $\lambda = -\omega^2$

$r = \pm\omega$ and $y(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}$

apply ICs:

$$\begin{cases} y(0) = 0 \\ y(\frac{\pi}{2}) = 0 \end{cases} \iff \begin{cases} 0 = c_1 + c_2 \\ 0 = c_1 e^{\frac{\omega\pi}{2}} + c_2 e^{-\frac{\omega\pi}{2}} \end{cases}$$

$$\iff \begin{cases} c_2 = -c_1 \\ 0 = c_1\left(e^{\frac{\omega\pi}{2}} - e^{-\frac{\omega\pi}{2}}\right) \end{cases}$$

$$\iff \begin{cases} c_2 = 0 \\ c_1 = 0 \end{cases}$$

$\therefore$ only trivial solutions.

ii) $\lambda > 0$ or $\lambda = \omega^2$

$r = \pm i\omega$ and $y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

apply ICs:

$$\begin{cases} y(0) = 0 \\ y(\frac{\pi}{2}) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 \sin(\frac{\pi}{2}\omega) = 0 \end{cases}$$

If $\frac{\pi}{2}\omega = n\pi$, $n \in \mathbb{Z}$, then c_2 arbitrary

else $c_2 = 0$.

Thus, we have

a) only trivial solutions if $\lambda < 0$ or if $\lambda > 0$, i.e. $\lambda = \omega^2$ and $\omega \neq 2n$, $n \in \mathbb{Z}$

b) nontrivial solutions if $\lambda > 0$, i.e. $\lambda = \omega^2$ and $\omega = 2n$, $n \in \mathbb{Z}$

(so $\lambda = 4n^2$)

4. PDE: $u_x + u_y = 0$

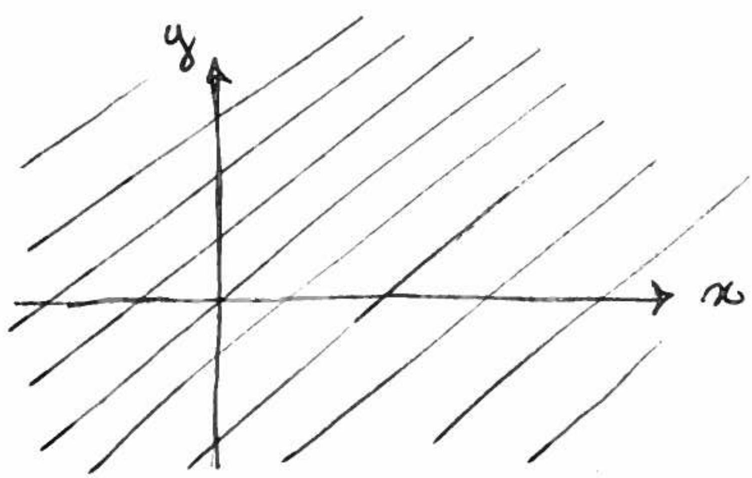
preferred direction: $\hat{i} + \hat{j} = (1,1)$

$\therefore$ the family of characteristic lines is given by

$$w = x - y$$

for different constant values of w. Thus, each value of w corresponds to a different line:

$$w = x - y \iff y = x - w$$



5. $u_x + u_y + u = e^{-3y}$ ... (1)

prefered direction: $\vec{m} = \hat{i} + \hat{j}$

$$\begin{cases} w = x - y \\ z = y \end{cases} \quad \text{or} \quad \begin{cases} x = w + z \\ y = z \end{cases} \quad \ldots (2)$$

Let $v(w,z) = u(x,y)$. Then $u_x + u_y = v_z$ and so eq. (1) becomes

$$v_z + v = e^{-3z} \quad \text{or} \quad \frac{dv}{dz} + v = e^{-3z} \quad \ldots (3)$$

Integrating factor:

$$\mu(z) = e^{\int 1\, dz} = e^{z} \quad \ldots\ldots (4)$$

Multiplying (3) by $\mu(z)$ we obtain

$$e^{z}\frac{dv}{dz} + e^{z}v = e^{-2z} \Leftrightarrow \frac{d}{dz}\left(e^{z}v\right) = e^{-2z} \Leftrightarrow$$

$$\Leftrightarrow e^{z}v = \int e^{-2z}dz = -\frac{1}{2}e^{-2z} + C(w)$$

$$\Leftrightarrow v = -\frac{1}{2}e^{-3z} + e^{-z}C(w) \quad \ldots (5)$$

Using (2) we rewrite (5) in terms of x & y:

$$u(x,y) = -\frac{1}{2} e^{-3y} + e^{-y} c(x+y).$$

This is the general solution.

6. $$\begin{cases} u_{tt} = 4u_{xx} & 0 < x < L,\ t > 0 \\ u(0,t) = u(4,t) = 0 \\ u(x,0) = 4\sin\left(\frac{3\pi x}{4}\right) \end{cases}$$

a) Let $v(x,t) = K\sin\left(\frac{3\pi x}{4}\right)\cos\left(\frac{6\pi t}{4}\right)$, and see if $u(x,t) = v(x,t)$ is a solution to the PDE:

$$LHS = v_{tt} = -K\sin\left(\frac{3\pi x}{4}\right)\left(\frac{6\pi}{4}\right)^2 \sin\left(\frac{6\pi t}{4}\right)$$

$$= \frac{-9K\pi^2}{4}\sin\left(\frac{3\pi x}{4}\right)\sin\left(\frac{6\pi t}{4}\right)$$

$$RHS = 4v_{xx} = 4K\left(-\left(\frac{9\pi}{4}\right)^2 \sin\left(\frac{3\pi x}{4}\right)\right)\cos\left(\frac{6\pi t}{4}\right)$$

$$= -\frac{9K\pi^2}{4}\sin\left(\frac{3\pi x}{4}\right)\cos\left(\frac{6\pi t}{4}\right)$$

$\because$ LHS = RHS, $v(x,t)$ is a solution of the PDE.

Now we test the BCs & IC (eq.(2)):

$$u(0,t) = v(0,t) = 0 \checkmark$$
$$u(4,t) = v(4,t) = 0 \checkmark$$
$$u(x,0) = v(x,0) = K \sin\left(\frac{3\pi x}{4}\right) = 4 \sin\left(\frac{3\pi x}{4}\right)$$

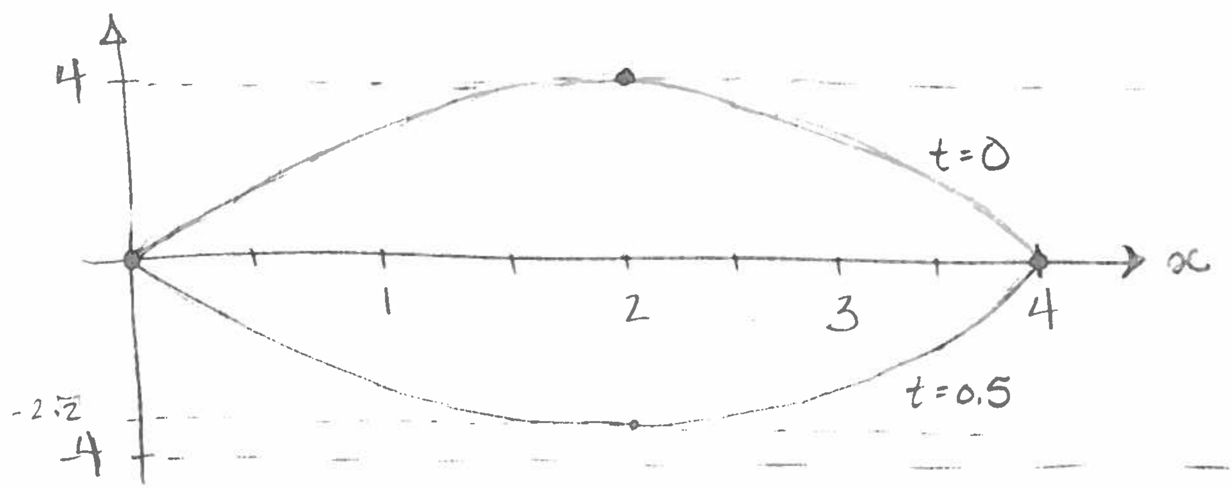
$\therefore$ $v(x,0)$ is a solution of (1) with (2) if $K=4$.

b) @ $t=0$, $u(x,0) = 4 \sin\left(\frac{3\pi x}{4}\right)$

@ $t=0.5$, $u(x,0) = 4 \sin\left(\frac{3\pi x}{4}\right) \cos\left(\frac{\cancel{6}^{3}\pi}{\cancel{4}_{2}} \frac{1}{2}\right)$

$$= 4 \sin\left(\frac{3\pi x}{4}\right) \cos\left(\frac{3\pi}{4}\right)$$

$$= -\frac{4\sqrt{2}}{2} \sin\left(\frac{3\pi x}{4}\right) = -2\sqrt{2} \sin\left(\frac{3\pi x}{4}\right)$$



7. $\because$ the functions $\sin(n\pi x)$, $n = 1,2,3,\ldots$, are all linearly independent, and $\because$ the LHS and RHS are each a linear combination of these same linearly independent functions, the coefficients on each side of the equation must match.

$$\therefore \quad b_2 = 5, \; b_5 = -7, \quad b_9 = 2$$

$$\text{and } \; b_n = 0 \quad \forall n \in \mathbb{N} \setminus \{2,5,9\}$$

(where $\mathbb{N} = \{1,2,3,4,\ldots\}$).