UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Mar 24th, 2025 Time: 8:00am Duration: 35 minutes.
This exam has 5 questions for a total of 28 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 25 minutes.

3 1. A 10 kg mass is attached to a spring with stiffness 10 N/m. There is essentially no friction (or damping). At what forcing frequency will the response of the system be largest? How will the forced system behave? What is this behaviour called?

No damping, $\therefore$ resonance frequency $= \omega = \sqrt{\frac{k}{m}} = 1$

The response will be largest when the forcing frequency is equal to ω.

This behaviour is called resonance.

2. Suppose that the equation of motion for a mass-spring system is given by

$$y(t) = e^{-t}(\cos(2t) + \sqrt{3}\sin(2t)) + 8(\cos(2t) - \sin(2t)). \quad (1)$$

(a) By looking at (1) (i.e., no calculations necessary), answer the following:

2 i. Is there friction present? Explain.

Yes. Friction gives rise to the decaying exponential e^{-t}.

2 ii. Is the mass-spring system being forced at the resonance frequency? Explain.

No, The natural frequency (2) and resonance frequency are not quite the same.

$\beta = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$, $\gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$

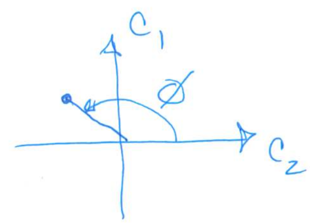
difference

3 (b) Express the steady-state part of the motion as a single phase-shifted sine. *Give an exact value for the phase shift.*

Steady state portion is $y_p(t)$:

$$y_p(t) = 8(\cos(2t) - \sin(2t)) = 8\cos(2t) - 8\sin(2t)$$

$$= A\sin(2t + \phi)$$



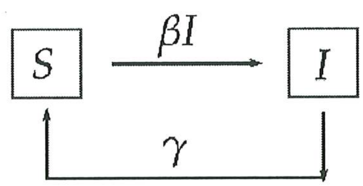
$$A = \sqrt{8^2 + 8^2} = \sqrt{2 \cdot 64} = 8\sqrt{2}$$

$$\tan(\phi) = \frac{8}{-8} = -1$$

$$\therefore \phi = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

$$\therefore y_p(t) = 8\sqrt{2}\sin\left(2t + \frac{3\pi}{4}\right)$$

3. Consider a disease where recovered individuals do not go into a recovered class, but immediately become susceptible again (so this would be a disease for which the body develops no immunity). The diagram for this disease is shown at left. Assume that $S + I = N$, where N is constant.



2 (a) Write down the two ODEs that describe this system.

$$\frac{dS}{dt} = -\beta IS + \gamma I$$

$$\frac{dI}{dt} = \beta IS - \gamma I$$

4 (b) Setting $u = S/N$, $v = I/N$, $t^* = \gamma t$, and $R_0 = \beta N/\gamma$, we have $u + v = 1$ and

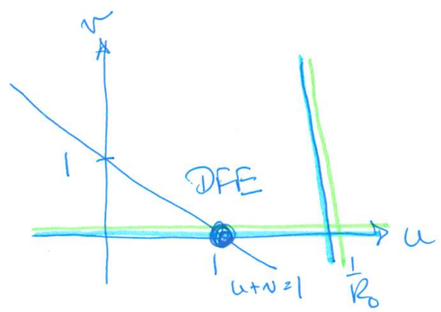
$$u' = -R_0 uv + v, \quad v' = R_0 uv - v. \tag{2}$$

Draw the (u, v) phase plane diagram and indicate the nullclines & steady states. Note that there are two cases! Label, when they exist, the disease-free equilibrium and the endemic equilibrium (EE).

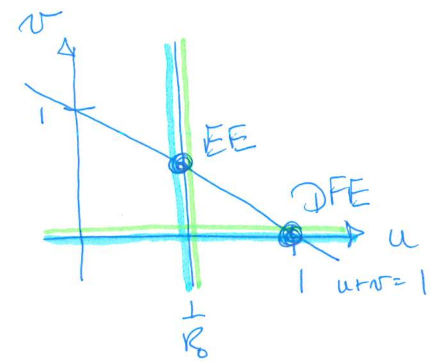
Nullclines:

$$\begin{cases} u' = 0 \\ v' = 0 \end{cases} \iff \begin{cases} -R_0 uv + v = 0 \\ R_0 uv - v = 0 \end{cases} \iff \begin{cases} v(1 - R_0 u) = 0 \\ v(R_0 u - 1) = 0 \end{cases} \Rightarrow \begin{cases} v = 0 \text{ or } u = \frac{1}{R_0} \\ v = 0 \text{ or } u = \frac{1}{R_0} \end{cases}$$

Case 1: $R_0 < 1$



Case 2: $R_0 > 1$



4 (c) Use the Jacobian to derive the condition under which the DFE is unstable.

$$J = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix} = \begin{bmatrix} -R_0 v & -R_0 u + 1 \\ R_0 v & R_0 u - 1 \end{bmatrix}$$

At the DFE we have $(u, v) = (1, 0)$.

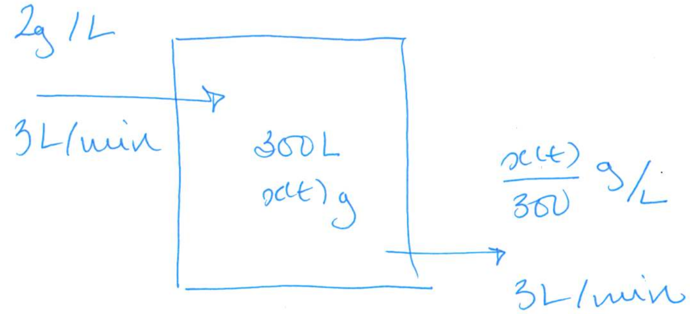
$$\therefore J\Big|_{DFE} = \begin{bmatrix} 0 & -R_0 + 1 \\ 0 & R_0 - 1 \end{bmatrix}$$

$\therefore \lambda_1 = 0, \; \lambda_2 = R_0 - 1$

For the DFE to be unstable we require $\lambda_2 > 0$

so $R_0 > 1$.

[8] 4. A large mixing tank initially holds 300 litres of brine. Another brine solution, with a salt concentration of 2 grams/litre is pumped into the tank at a rate of 3 litres per minute. The mixture in the tank is well-stirred, and is pumped out at the same rate (3 litres per minute). If the tank initially contained 50 grams of salt, how much salt is in the tank at time t?



$x(0) = 50$ g

We want $x(t)$.

Solution:

$$\frac{dx}{dt} = 2\frac{g}{\cancel{L}} \cdot 3\frac{\cancel{L}}{min} - \frac{x(t)}{300}\frac{g}{\cancel{L}} \cdot 3\frac{\cancel{L}}{min}$$

The units all match, so we can drop them. The ODE is thus

$$\frac{dx}{dt} = 6 - \frac{1}{100}x \quad \ldots (3)$$

We can solve using an integrating factor. Writing (3) in standard form we obtain

$$(4) \quad \ldots \frac{dx}{dt} + \frac{1}{100}x = 6 \qquad \therefore \mu(t) = e^{\int \frac{1}{100} dt} = e^{\frac{1}{100}t}$$

Multiplying (4) by $\mu(t)$:

$$e^{\frac{1}{100}t}\frac{dx}{dt} + \frac{1}{100}e^{\frac{1}{100}t}x = 6e^{\frac{1}{100}t} \iff \frac{d}{dt}\left(e^{\frac{1}{100}t}x\right) = 6e^{\frac{1}{100}t} \iff \int$$

$$\iff e^{\frac{1}{100}t}x = 600e^{\frac{1}{100}t} + C \iff x(t) = 600 + Ce^{-\frac{1}{100}t}$$

Apply the IC: $x(0) = 50 \iff 50 = 600 + C \iff C = -550$

$$\therefore \boxed{x(t) = 600 - 550e^{-\frac{1}{100}t}}$$

☐ 5. **BONUS (3 pts)** In the case where the DFE is unstable (question 3(c)), what is the stability of the EE?

The EE $= \left(\frac{1}{R_0}, 1-\frac{1}{R_0}\right)$.

So $J\Big|_{EE} = \begin{bmatrix} -R_0\left(1-\frac{1}{R_0}\right) & -R_0\frac{1}{R_0}+1 \\ R_0\left(1-\frac{1}{R_0}\right) & R_0\left(\frac{1}{R_0}\right)-1 \end{bmatrix} = \begin{bmatrix} -R_0+1 & 0 \\ R_0-1 & 0 \end{bmatrix}$

$\therefore \lambda_1 = -R_0+1 = 1-R_0 < 0$

$\lambda_2 = 0$

$\because$ neither λ_1 nor λ_2 is positive, the EE is stable.