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Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Mar 3rd, 2025 Time: 8:00am Duration: 35 minutes.
This exam has 5 questions for a total of 23 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

$\boxed{4}$ 1. Find the general solution of the differential equation $4y'' - 4y' + y = 0$.

char eqn : $4r^2 - 4r + 1 = 0 \Leftrightarrow r = \dfrac{2 \pm \sqrt{4-4}}{4} = \dfrac{1}{2}$

$\therefore y(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$

$\boxed{5}$ 2. Find a particular solution to

$$y'' + 2y' + 2y = \cos(t),$$

Given that a fundamental solution set for the ODE is $\{e^{-t}\sin(t), e^{-t}\cos(t)\}$.

We can use MOUC: Try $y_p(t) = A\cos(t) + B\sin(t)$.

Then $y_p'(t) = -A\sin(t) + B\cos(t)$

$y_p''(t) = -A\cos(t) - B\sin(t)$

Plugging y_p + its derivatives into the ODE:

$-A\cos(t) - B\sin(t) + 2(-A\sin(t) + B\cos(t))$

$+ 2(A\cos(t) + B\sin(t)) = \cos(t) \Leftrightarrow$ (1)

(1) $\Leftrightarrow \begin{cases} -A + 2B + 2A = 1 \\ -B - 2A + 2B = 0 \end{cases} \Leftrightarrow \begin{cases} A = 1 - 2B \\ B = 2 - 4B \end{cases} \Leftrightarrow \begin{cases} A = 1/5 \\ B = 2/5 \end{cases}$

$\therefore y_p(t) = \frac{1}{5}\cos(t) + \frac{2}{5}\sin(t)$

3 3. Determine the equation ~~of~~ of motion, $y(t)$, for a mass-spring system given the following information:

(i). $\{e^t, te^t\}$ is a fundamental solution set for the system,

(ii). $(t^2/2)e^t$ is a particular solution for the system, and

(iii). $y(0) = 0$, $y'(0) = 3$ are the initial conditions for the system.

$$y(t) = c_1 e^t + c_2 te^t + \frac{t^2}{2} e^t$$

Apply the ICs: $\begin{cases} 0 = c_1 \\ 3 = c_1 + c_2 \end{cases} \iff \begin{matrix} c_1 = 0 \\ c_2 = 3 \end{matrix}$

$$\therefore \; y(t) = 3te^t + \frac{t^2}{2} e^t$$

4 4. Express the particular solution to the initial value problem

$$y'' - y = \frac{1}{t}, \qquad y(1) = 0, \quad y'(1) = -2,$$

using integrals (do not attempt to solve these!). The fundamental solution set is $\{e^t, e^{-t}\}$.

Try $y_p(t) = v_1(t)e^t + v_2(t)e^{-t}$. Then v_1 & v_2 must satisfy

$$\begin{cases} v_1' e^t + v_2' e^{-t} = 0 \\ v_1' e^t - v_2' e^{-t} = \frac{1}{t} \end{cases} \iff \begin{cases} 2v_1' e^t = \frac{1}{t} \\ 2v_2' e^{-t} = -\frac{1}{t} \end{cases} \iff \begin{cases} v_1' = \frac{e^{-t}}{2t} \\ v_2' = -\frac{e^t}{2t} \end{cases}$$

$$\therefore \; y_p(t) = \left(\int \frac{e^{-t}}{2t}\,dt\right)e^t - \left(\int \frac{e^t}{2t}\,dt\right)e^{-t}$$

5. A 1/4 kg mass is attached to a spring with stiffness 8 N/m. The damping constant for the system is 1/4 Ns/m. There is no forcing.

$\boxed{1}$ (a) Write down the differential equation that describes the motion of the mass.

$$\frac{1}{4}y'' + \frac{1}{4}y' + 8y = 0$$

$\boxed{1}$ (b) Suppose the mass is moved 1 m to the left of equilibrium and released. What are the corresponding initial conditions?

$$y(0) = -1 \; ; \; y'(0) = 0$$

$\boxed{3}$ (c) Determine the quasiperiod and exponential decay constant of the motion.

char eqn: $\frac{1}{4}r^2 + \frac{1}{4}r + 8 = 0 \iff r^2 + r + 32 = 0$

$\therefore r = \frac{-1 \pm \sqrt{1-128}}{2} = \frac{-1 \pm \sqrt{-127}}{2} = -\frac{1}{2} \pm \frac{\sqrt{127}}{2} i$

$\therefore$ quasiperiod is $\frac{2\pi}{\sqrt{127}/2} = \frac{4\pi}{\sqrt{127}}$, exp'l decay cst is $-\frac{1}{2}$

$\boxed{2}$ (d) Sketch the motion. Clearly indicate the envelope, the initial conditions, and the oscillatory behaviour. What is the equation for the envelope? *(You do not need to solve the equation of motion to answer this question!)*