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THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Jan 29th, 2025 Time: 8:40am Duration: 20 minutes.
This exam has 6 questions for a total of 42 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

1. Consider the ODE $y' + 3ty = 2t$.

$\boxed{3}$ (a) Classify the ODE.

linear, 1^{st} order

$\boxed{6}$ (b) Solve the ODE.

Integrating factor $\mu(t) = e^{\int 3t\,dt} = e^{3t^2/2}$

The ODE becomes

$$e^{3t^2/2}\frac{dy}{dt} + 3te^{3t^2/2}y = 2te^{3t^2/2} \Leftrightarrow \frac{d}{dt}\left(e^{3t^2/2}y\right) = 2te^{3t^2/2} \Leftrightarrow \int$$

$$\int \Leftrightarrow e^{3t^2/2}y = \int \frac{2}{3}3te^{3t^2/2}\,dt \qquad \text{let } u = \frac{3t^2}{2} \quad du = 3t\,dt$$

$$\Leftrightarrow e^{3t^2/2}y = \frac{2}{3}\int e^u\,du = \frac{2}{3}e^u + C = \frac{2}{3}e^{3t^2/2} + C$$

$$\Leftrightarrow y = \frac{2}{3} + Ce^{-3t^2/2}$$

$\boxed{2}$ (c) Suppose the initial condition is $y(0) = 1$. Solve for the constant of integration and find the unique solution.

$$y(0) = 1 \Leftrightarrow 1 = \frac{2}{3} + C \Leftrightarrow C = \frac{1}{3}$$

$$\therefore y(t) = \frac{2}{3} + \frac{1}{3}e^{-3t^2/2}$$

[8] 2. Verify that the equation below is exact, and solve it.

$$(x\cos(y))dy + \left(\overset{2x+\sin(y)}{\cancel{2y - \frac{x^2}{2}\sin(y)}}\right) dx = 0$$

Step1
Test for exactness:

$$\frac{\partial}{\partial x}(x\cos(y)) = \cos(y) = \frac{\partial}{\partial y}(2x+\sin(y)) = \cos(y)$$

$\because$ the two derivatives are equal, the ODE is exact.

step2: Integrate wrt x

$$F(x,y) = \int (2x+\sin(y))\,dx = x^2 + x\sin(y) + h(y)$$

step3: Differentiate wrt y

$$\frac{\partial F}{\partial y} = x\cos(y) \iff x\cos(y) + h'(y) = x\cos(y)$$

$$\iff h'(y) = 0 \iff h(y) = \cancel{C}\,0$$

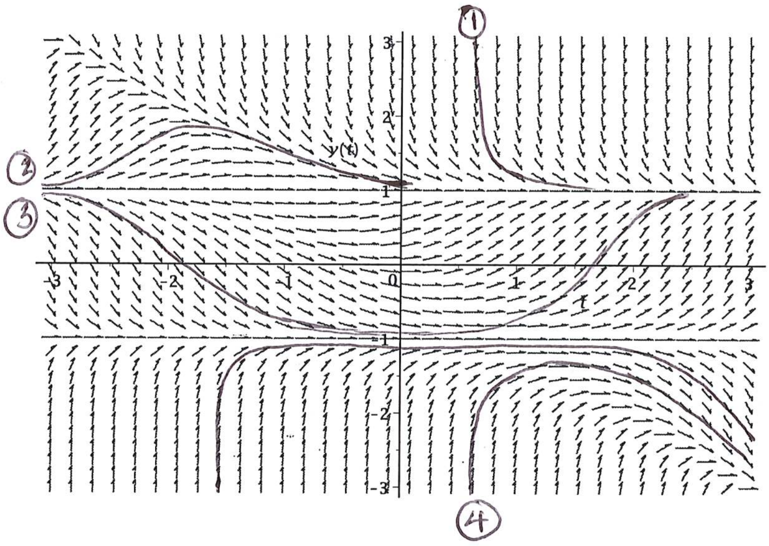
step4: Form F

$$F(x,y) = x^2 + x\sin(y) + 0 = x^2 + x\sin(y)$$

step5: The solutions are the level curves of F:

$$\boxed{x^2 + x\sin(y) = C}$$

3. Use the given direction field to answer the questions below.



$\boxed{4}$ (a) On the direction field, sketch several solution curves showing the different solution behaviours that can occur (there are 4 distinct behaviours).

$\boxed{2}$ (b) In general, give conditions on t and y for which solutions are bounded as $t \to \infty$. In the cases where $y(t)$ is bounded, what is $\lim_{t\to\infty} y(t)$?

Solutions are bounded $\forall\, y(t_0) \geq -1$.

If $y_0 > -1$, $\lim\limits_{t\to\infty} y(t) = 1$

$y_0 = -1$, $\lim\limits_{t\to\infty} y(t) = -1$

$\boxed{4}$ (c) Note that zero slopes are found along the line $y = -t$ and also along the lines $y = \pm 1$. Write the equation for a first order ODE that could have a direction field like this.

$$\frac{dy}{dt} = -(y+t)(y^2-1)$$

$\boxed{5}$ 4. Find the solution of

$$\frac{dy}{dx} = -xy^2, \quad y(0) = \frac{1}{2}.$$

$$\frac{dy}{dx} = -xy^2 \Rightarrow \int \frac{dy}{y^2} = \int -x\,dx \Leftrightarrow -\frac{1}{y} = -\frac{x^2}{2} + C \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{y} = \frac{x^2}{2} - \frac{\tilde{C}}{2} \Leftrightarrow y = \frac{2}{x^2 - \tilde{C}}$$

Apply the IC:

$$y(0) = \frac{1}{2} \Leftrightarrow \frac{1}{2} = \frac{2}{-\tilde{C}} \Leftrightarrow \frac{1}{2} = \frac{-2}{\tilde{C}} \Leftrightarrow \tilde{C} = -4$$

$$\therefore \boxed{y(x) = \frac{2}{x^2+4}}$$

$\boxed{8}$ 5. Consider the initial value problem

$$y' = -ty, \qquad y(0) = 2.$$

Find $y(1)$ using Euler's method with stepsize $h = 0.5$. Place your answers in the table below, and use the space below the table to show your work. Decimal answers should be accurate to 2 decimal places. Circle the value corresponding to your approximation for $y(1)$.

n	t_n	y_n	f_n	y_{n+1}
0	0	2	0	2
1	0.5	2	-1	1.5
2	1	1.5		

$y_{n+1} = y_n + h f(t_n, y_n)$

So:

$y_1 = y_0 + h(-t_0 y_0) = 2 + 0.5(0) = 2$

$y_2 = y_1 + h(-t_1 y_1) = 2 + 0.5\left(-\frac{1}{2} \cdot 2\right) = 2 + \frac{1}{2}(-1)$

$= \frac{3}{2} = 1.5$

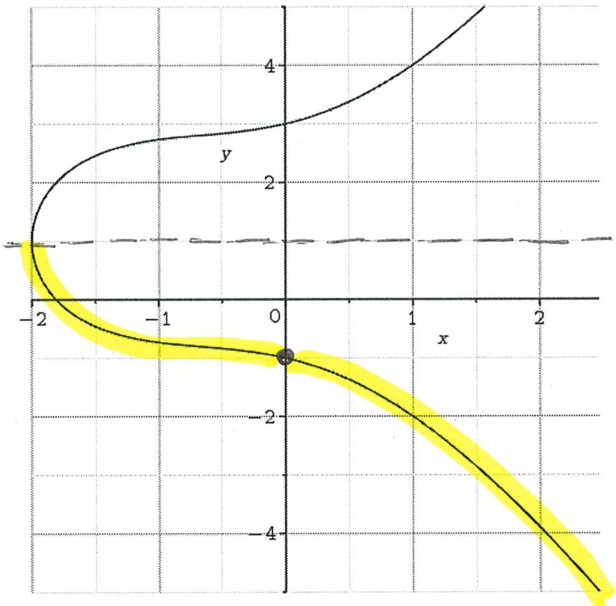
☐ 6. **BONUS (3 pts)** (*Drawn from a discussion we had in the review session Monday morning.*) Consider the IVP

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1. \tag{1}$$

Using separation of variables we obtain the implicit solution

$$y^2 + 2y = x^3 + 2x^2 + 2x + 3. \tag{2}$$

If we plot (2) we obtain the curve shown in Figure 1.



solution curve —

IC •

singularity ---

Figure 1: Plot of (2).

Explain why the solution to the IVP (1) does NOT consist of the complete curve in Figure 1. Indicate which part of the curve IS the solution.

(1) Has a singularity at $y = 1$, so only the portion of the curve <u>below</u> $y = 1$, & which includes the IC $y(0) = -1$, is the solution to the IVP.