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Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Apr 17th, 2025 Start Time: 8:30am **End Time: 11:00am**
Duration: 2.5 hours
This exam has **8 questions** for a total of **93 points**.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **There will be no marks for answers without supporting work.** Simplify all answers.

You have 2.5 hours to complete the exam individually. There is no group portion for this exam.

Question:	1	2	3	4	5	6	7	8	Total
Points:	17	11	14	6	6	13	10	3	80
Score:									

1. Consider the ODE

$$\frac{dy}{dx} = x^2 - y^2. \tag{1}$$

The direction field for the ODE is shown in Figure 1.

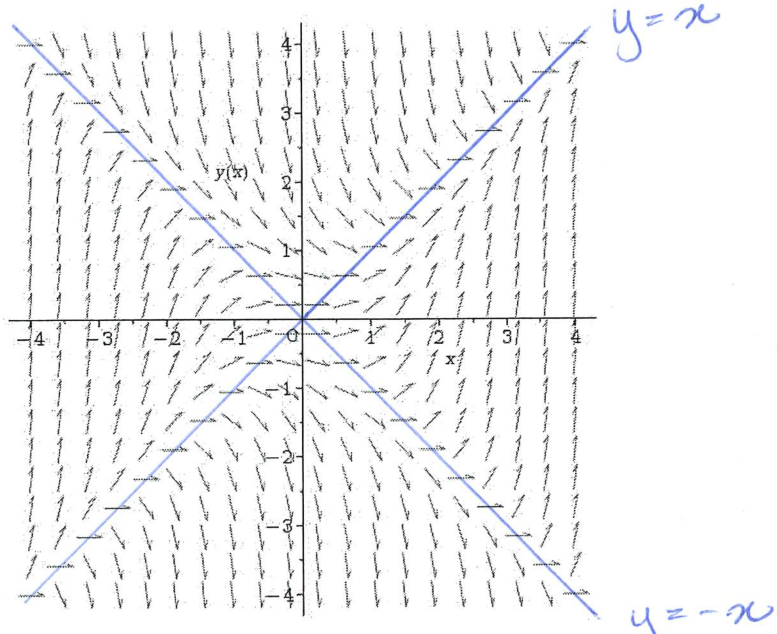


Figure 1: Direction field for question 1.

$\boxed{3}$ (a) Solve for the zero isoclines of (1) and draw them on Figure 1 (above). Are the direction field arrows that cross your zero isoclines pointing the right way? Explain.

$$\frac{dy}{dx} = 0 \iff x^2 - y^2 = 0 \iff y^2 = x^2 \iff y = \pm x$$

$\frac{dy}{dx} = 0$ means that y is not changing with x, So flow should be horizontal. We see that indeed the flow arrows are all $\rightarrow$ along the two zero isoclines.

$\boxed{2}$ (b) Rewrite the ODE using the Forward Euler method and solve for y_{n+1}.

$$\frac{y_{n+1} - y_n}{h} = x_n^2 - y_n^2 \iff y_{n+1} = y_n + h(x_n^2 - y_n^2)$$

(b) Below is a plot of the direction field in the first quadrant. Explore the solution with the initial condition $y(0) = 1$ as follows.

$\boxed{2}$ i. Sketch the solution by eye on the direction field.

$\boxed{2}$ ii. Carefully plot the Forward Euler solution using a stepsize of $h = 1$.

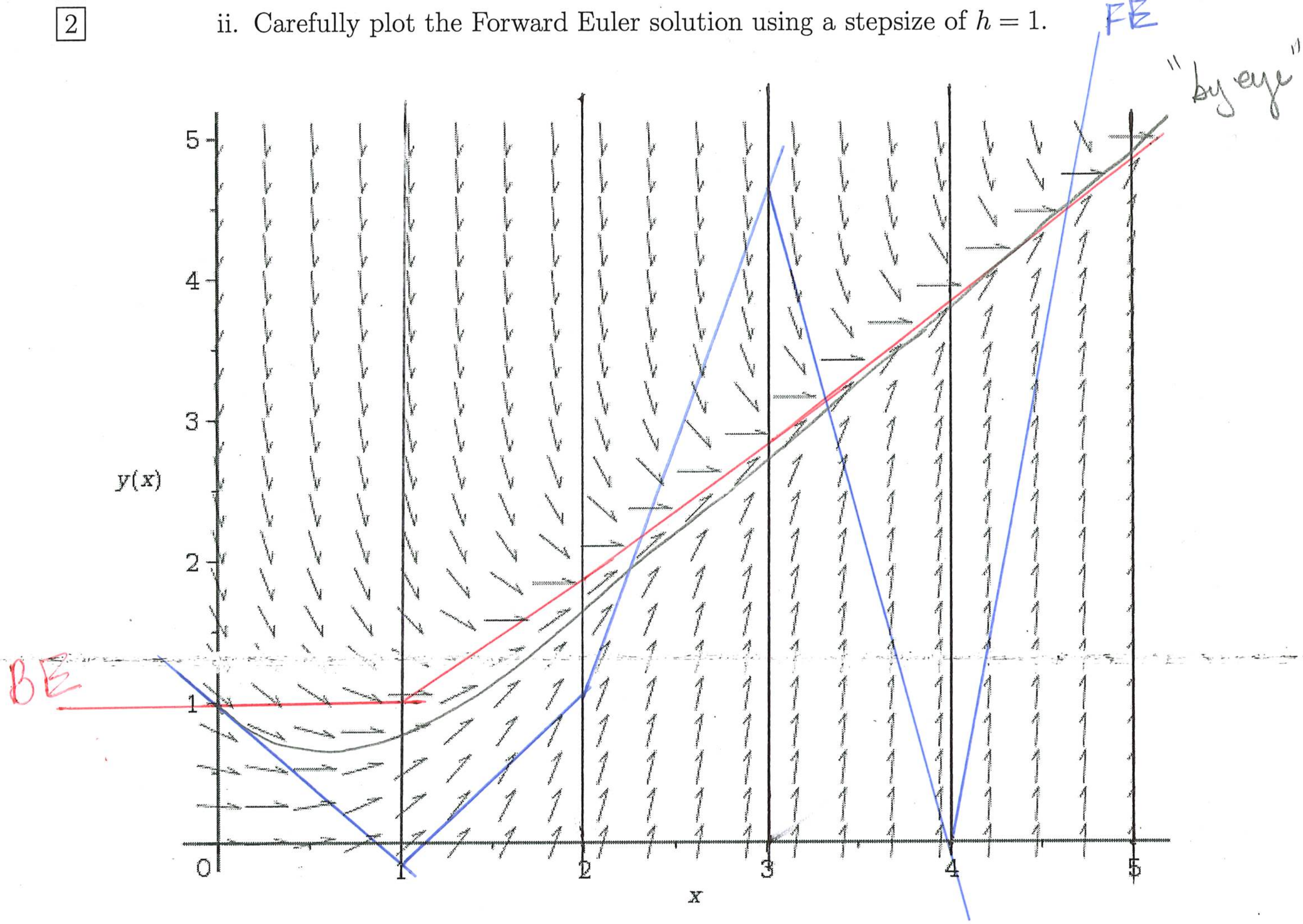


Figure 2: Direction field for question 1.

$\boxed{2}$ (c) Why do the "by eye" and Forward Euler solutions differ so much? Explain, with reference to the structure of the Forward Euler method.

The FE solution uses the information only at the beginning of each timestep, while the "by eye" solution is continually adjusted as the slope field changes.

1 (e) After $x = 2$, the FE method starts to demonstrate problematic behaviour. What is this behaviour called?

Grid-scale oscillations.

2 (f) What first order method could you use to obtain a solution that follows the direction field without generating the observed problematic behaviour?

1 i. Name that method: Backward Euler

2 ii. Using this method, plot the solution on the direction field (Figure 2) using a stepsize of $h = 1$ (use a different colour and label all three solutions!).

2 iii. Rewrite the ODE using this method.

$$\frac{y_{n+1} - y_n}{h} = x_{n+1}^2 - y_{n+1}^2$$

2. A 500 litre mixing tank initially holds 300 litres of fresh water. A brine solution, with a salt concentration of 2 grams/litre is pumped into the tank at a rate of 3 litres per minute. The mixture in the tank is well-stirred, and is pumped out at a rate of 1 litre per minute.

[2] (a) When will the tank overflow?

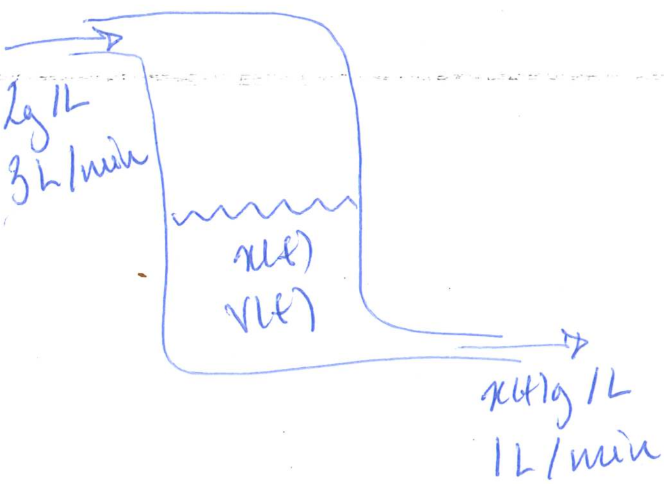
$V(t) = 300 + 2t$

Overflow at

$V(t) = 500 \iff 300 + 2t = 500 \iff \boxed{t = 100 \text{ minutes}}$

[3] (b) Show that the amount of salt in the tank at time t, $x(t)$, is governed by the equation below. Include a sketch.

$$\frac{dx(t)}{dt} + \frac{1}{300 + 2t}\,x(t) = 6$$



$$\frac{dx}{dt} = \text{inflow} - \text{outflow}$$

$$= 2\frac{g}{L} \times 3\frac{L}{min} - \frac{x(t)}{V(t)\,L} \times 1\frac{L}{min}$$

$$\iff \frac{dx}{dt} = 6 - \frac{x}{300+2t} \iff \frac{dx}{dt} + \frac{x}{300+2t} = 6$$

As required.

6 (c) If the tank initially contained 50 grams of salt, how much salt is in the tank when it overflows?

① integrating factor

$$\mu(t) = e^{\int \frac{1}{300+2t}\,dt} = e^{\frac{1}{2}\int \frac{1}{150+t}\,dt} = e^{\frac{1}{2}\ln(150+t)}$$

$$= e^{\ln(150+t)^{1/2}} = \sqrt{150+t}$$

② multiply the ODE by $\mu(t)$ + integrate

$$\int \frac{d}{dt}\left(\mu(t)\,x(t)\right)dt = \int 6\,\mu(t)\,dt \iff$$

$$\iff \sqrt{150+t}\,x(t) = 6\int (150+t)^{1/2}\,dt$$

$$\iff \sqrt{150+t}\,x(t) = 6\left(\frac{2}{3}\right)(150+t)^{3/2} + C$$

$$\iff x(t) = 4(150+t) + \frac{C}{\sqrt{150+t}}$$

③ apply the IC

$$x(0) = 50 \iff 50 = 4(150) + \frac{C}{\sqrt{150}} \iff -550 = \frac{C}{5\sqrt{6}}$$

$$\iff C = -2750\sqrt{6}$$

④ evaluate $x(100)$

$$\boxed{x(100) = 4(150+100) - \frac{2750\sqrt{6}}{\sqrt{150+100}}} = 1000 - \frac{\cancel{2750}^{550}\sqrt{6}}{\cancel{5}\sqrt{10}}$$

$$= 1000 - 550\sqrt{\frac{3}{5}} = \boxed{50\left(40 - 11\sqrt{\frac{3}{5}}\right) \text{ grams}}$$

3. Consider the differential equation $y'' - 4y' + 4y = e^{2t}$.

3 (a) Find the homogeneous solution.

① char. eqn.

$r^2 - 4r + 4 = 0 \Leftrightarrow (r-2)^2 = 0 \Leftrightarrow r = 2$

$\therefore y_h(t) = c_1 e^{2t} + c_2 t e^{2t}$

5 (b) Use the method of undetermined coefficients to derive a particular solution.

② Let $y_p(t) = At^2e^{2t}$

Then $y_p'(t) = 2Ate^{2t} + 2At^2e^{2t}$

$y_p''(t) = 2Ae^{2t} + 4Ate^{2t} + 4Ate^{2t} + 4At^2e^{2t}$

$= 2Ae^{2t} + 8Ate^{2t} + 4At^2e^{2t}$

③ Plug $y_p(t)$ into the ODE

$y_p'' - 4y_p' + 4y_p = e^{2t} \Leftrightarrow 2Ae^{2t} + 8Ate^{2t} + 4At^2e^{2t} - 4(2Ate^{2t} + 2At^2e^{2t}) + 4At^2e^{2t} = e^{2t} \Leftrightarrow (\text{I})$

$(\text{I}) \Leftrightarrow e^{2t}\left(2A + \cancel{8At} + \cancel{4At^2} - \cancel{8At} - \cancel{8At^2} + \cancel{4At^2}\right) = e^{2t}$

$\Leftrightarrow 2A = 1 \Leftrightarrow A = \frac{1}{2}$

$\therefore \boxed{y_p(t) = \frac{1}{2}t^2e^{2t}}$

5 (c) Use the method of variation of parameters to derive a particular solution. Verify that you obtain the same particular solution. *Your* solution must start with the two constraints that are derived from the variation of parameters assumption.

(4) Let $y_p(t) = v_1(t)e^{2t} + v_2(t)te^{2t}$. Then

$$\begin{cases} v_1' e^{2t} + v_2' te^{2t} = 0 \\ v_1' 2e^{2t} + v_2'(e^{2t} + 2te^{2t}) = e^{2t} \end{cases} \iff (1)$$

$$(1) \iff \begin{cases} v_1' + t v_2' = 0 \\ 2v_1' + (1+2t)v_2' = 1 \end{cases} \iff \begin{cases} v_1' = -t \\ v_2' = 1 \end{cases}$$

$$\iff \begin{cases} v_1 = -\frac{t^2}{2} \\ v_2 = +t \end{cases}$$

$$\therefore \boxed{y_p(t) = -\frac{t^2}{2}e^{2t} + t \cdot te^{2t} = \left(-\frac{t^2}{2} + t^2\right)e^{2t} = \frac{t^2}{2}e^{2t}}$$

This is the same as the answer obtained in (b).

1 (d) Form the general solution.

$$y(t) = c_1 e^{2t} + c_2 te^{2t} + \frac{t^2}{2}e^{2t}$$

$\boxed{6}$ 4. Consider the forced mechanical oscillator

$$y'' + by' + y = 5\sin(2t), \quad b \leq 2. \tag{1}$$

(a) In the case $b > 0$, what are the forcing frequency, the natural frequency, and the resonance frequency of this system?

char equ: $r^2 + br + 1 = 0 \Rightarrow r = \frac{-b \pm \sqrt{b^2-4}}{2} = -\frac{b}{2} \pm \frac{\sqrt{4-b^2}}{2} i$ $\because b \leq 2$

Natural Frequency $= \beta = \frac{\sqrt{4-b^2}}{2}$

Resonance Frequency $= \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} = \sqrt{\frac{1}{1} - \frac{b^2}{2}} = \sqrt{\frac{2-b^2}{2}}$

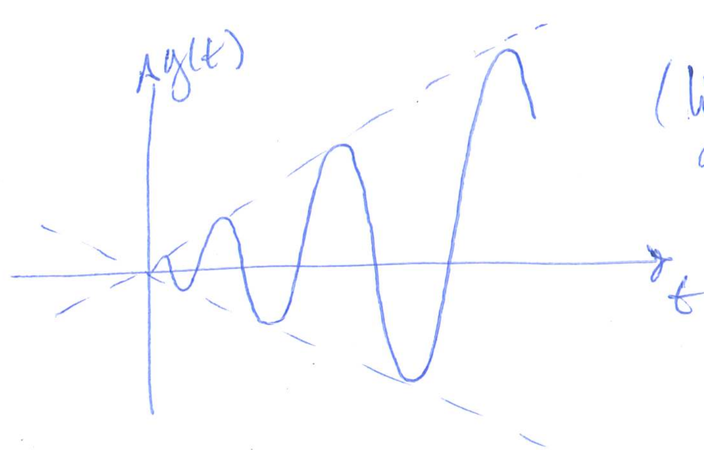
Forcing Frequency $= 2$

(b) In the case where $b = 0$, what forcing frequency will generate the largest response? Sketch the behaviour.

When $b=0$, the resonance frequency is

$\omega = \sqrt{\frac{k}{m}} = 1.$

If the forcing frequency is equal to ω, the behaviour will be:



(linearly increasing amplitude)

5. Consider a mass-spring system whose equation of motion is given by

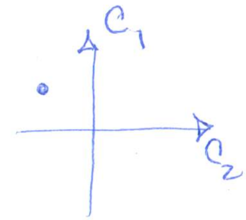
$$y(t) = e^{-3t}\left(\sqrt{3}\cos\left(\frac{t}{2}\right) - \sin\left(\frac{t}{2}\right)\right). \tag{2}$$

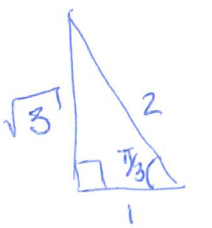
$\boxed{3}$ (a) Write the behaviour as a single phase-shifted sine.

$$y(t) = Ae^{-3t}\sin\left(\frac{t}{2}+\phi\right)$$

where $A = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\tan(\phi) = \dfrac{\sqrt{3}}{-1}$ 2nd quadrant





$\therefore \phi = \arctan(-\sqrt{3}) + \pi = -\dfrac{\pi}{3} + \pi = \dfrac{2\pi}{3}$

$$\therefore \boxed{y(t) = 2e^{-3t}\sin\left(\frac{t}{2}+\frac{2\pi}{3}\right)}$$

$\boxed{3}$ (b) When are the first two crossings of the equilibrium position at $y = 0$? *Note that* $\arctan(\sqrt{3}) = \pi/3$.

crossings occur when

$$y(t) = 0 \iff \sin\left(\frac{t}{2}+\frac{2\pi}{3}\right) = 0 \iff \frac{t_n}{2} + \frac{2\pi}{3} = n\pi, \quad n \in \mathbb{Z}$$

$$\iff \frac{t_n}{2} = -\frac{2\pi}{3} + n\pi \iff t_n = 2\pi\left(n - \frac{2}{3}\right)$$

So the first two crossings occur at

$$\boxed{t_1 = \frac{2\pi}{3}} \quad \text{and} \quad \boxed{t_2 = \frac{8\pi}{3}}$$

6. Consider a disease model with four classes, S, E, I, and R, where exposed individuals are all infectious (see Figure 3). Assume that the transmission rate of **I** individuals is higher than that of **E** individuals (i.e. $\beta > \alpha$).

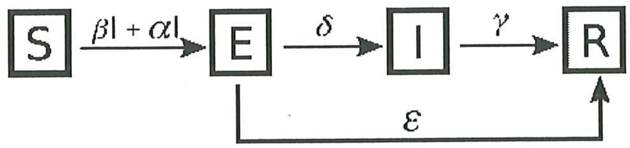


Figure 3: Compartmental diagram for the disease of problem 6.

4 (a) Write down the system of ODEs corresponding to the compartmental diagram.

$$\begin{cases} \dfrac{dS}{dt} = -(\beta+\alpha)IS \\[2ex] \dfrac{dE}{dt} = (\beta+\alpha)IS - (\delta+\varepsilon)E \\[2ex] \dfrac{dI}{dt} = \delta E - \gamma I \\[2ex] \dfrac{dR}{dt} = \gamma I + \varepsilon E \end{cases}$$

Note: Some of you noticed that the verbal description & diagram don't match. The ODEs above are for the diagram.

5 (b) In dimensionless terms, the model becomes

$$
\begin{aligned}
s' &= -R_0\, is, & &= f(s,e,i) \quad (3a)\\
e' &= R_0\, is - (\delta^* + \epsilon^*)\, e, & &= g(s,e,i) \quad (3b)\\
i' &= \delta^*\, e - i, & &= h(s,e,i) \quad (3c)\\
r' &= \epsilon^*\, e + i, & & \quad (3d)
\end{aligned}
$$

where the basic reproduction ratio is given by

$$R_0 = \frac{(\beta + \alpha)N}{\gamma}.$$

Since r only appears in the last equation, we can ignore it and focus on the first three equations. Find the Jacobian of this third order system and evaluate it at the disease-free equilibrium (DFE).

$$J = \begin{bmatrix} \frac{\partial f}{\partial s} & \frac{\partial f}{\partial e} & \frac{\partial f}{\partial i} \\ \frac{\partial g}{\partial s} & \frac{\partial g}{\partial e} & \frac{\partial g}{\partial i} \\ \frac{\partial h}{\partial s} & \frac{\partial h}{\partial e} & \frac{\partial h}{\partial i} \end{bmatrix} = \begin{bmatrix} -R_0 i & 0 & -R_0 s \\ R_0 i & -(\delta^* + \epsilon^*) & R_0 s \\ 0 & \delta^* & -1 \end{bmatrix}$$

The DFE $= (1,0,0)$ and so

$$\boxed{J\Big|_{DFE} = \begin{bmatrix} 0 & 0 & -R_0 \\ 0 & -(\delta^* + \epsilon^*) & R_0 \\ 0 & \delta^* & -1 \end{bmatrix}}$$

4 (c) The three eigenvalues of the Jacobian at the DFE are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \frac{1}{2}\left(-(\delta^* + \epsilon^* + 1) \pm \sqrt{(\delta^* + \epsilon^* + 1)^2 - 4(\delta^* + \epsilon^* - R_0\delta^*)}\right), \quad (4)$$

where $\delta^* = \delta/\gamma$ and $\epsilon^* = \epsilon/\gamma$. Using this information, determine the conditions on R_0 for the DFE to be unstable.

Hint: Look carefully at the eigenvalues before diving into the calculations. The calculations you need to do are short and simple!

Let $\delta^* + \varepsilon^* + 1 = B > 0$. Then

$$\lambda_{2,3} = \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4(B - (R_0\delta^* + 1))}\right)$$

Clearly $Re(\lambda_3) < 0$ so the only eigenvalue that could be positive is λ_2. This eigenvalue will be positive if

$$B - (R_0\delta^* + 2) < 0 \iff R_0\delta^* + 1 > B \iff R_0 > \frac{B-1}{\delta^*}$$

$$\iff R_0 > \frac{\delta^* + \varepsilon^*}{\delta^*} = 1 + \frac{\varepsilon^*}{\delta^*} = 1 + \frac{\varepsilon}{\delta}$$

$\therefore$ if $\boxed{R_0 > 1 + \frac{\varepsilon}{\delta}}$, the DFE will be unstable + the disease can invade.

$\boxed{10}$ 7. Using the method of Laplace Transforms, solve the initial value problem

$$y'' - 5y' + 6y = 12t, \qquad y(0) = 1, \quad y'(0) = 0.$$

When applying the Laplace or inverse Laplace transform, indicate at each step which properties you used (properties are listed on the last page).

(Extra workspace on the next page.)

① Apply the Laplace transform

$$\underbrace{\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\}}_{P1 \,\&\, P2} = \underbrace{12\mathcal{L}\{t\}}_{P2} \iff$$

$$\iff \underbrace{s^2Y(s) - s\overset{1}{y(0)} - \overset{0}{y'(0)}}_{P5} - 5[\underbrace{sY(s) - \overset{1}{y(0)}}_{P4}] + 6Y(s) = \underbrace{\frac{12}{s^2}}_{\text{Table}}$$

$$\iff (s^2 - 5s + 6)Y(s) - s + 5 = \frac{12}{s^2}$$

$$\iff Y(s) = \left(\frac{12}{s^2} + s - 5\right)\frac{1}{s^2 - 5s + 6} = \frac{12 + s^3 - 5s^2}{s^2(s-3)(s-2)}$$

② Partial fractions

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-2}$$

$$= \frac{As(s-3)(s-2) + B(s-3)(s-2) + Cs^2(s-2) + Ds^2(s-3)}{s^2(s-3)(s-2)}$$

$$= \frac{A(s^3 - 5s^2 + 6s) + B(s^2 - 5s + 6) + C(s^3 - 2s^2) + D(s^3 - 3s^2)}{s^2(s-3)(s-2)}$$

$$= \frac{s^3(A + C + D) + s^2(-5A + B - 2C - 3D) + s(6A - 5B) + 6B}{s^2(s-3)(s-2)}$$

(Extra workspace for question# 2.)

Comparing numerators we obtain

$$\begin{cases} A+C+D=1 \\ -5A+B-2C-3D=-5 \\ 6A-5B=0 \\ 6B=12 \end{cases} \Leftrightarrow \begin{cases} C+D=1-\frac{5}{3}=-\frac{2}{3} \\ 2C+3D=5-\frac{25}{3}+2=-\frac{4}{3} \\ A=5/3 \\ B=2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} C=-2/3 \\ D=-\frac{4}{3}+\frac{4}{3}=0 \\ A=5/3 \\ B=2 \end{cases}$$

$$\therefore Y(s)=\frac{5}{3s}+\frac{2}{s^2}-\frac{2}{3(s-3)}$$

③ Take the inverse Laplace transform

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

by linearity of $\mathcal{L}^{-1}$

$$=\frac{5}{3}+2t-\frac{2}{3}e^{3t}$$

from the Table.

So $\boxed{y(t)=\frac{5}{3}+2t-\frac{2}{3}e^{3t}}$

$\boxed{3}$ 8. Express the function below using window and step functions and compute its Laplace transform.

$$g(t) = \begin{cases} 0 & 0 < t < \frac{\pi}{2} \\ \sin(t) & t > \frac{\pi}{2} \end{cases}$$

$g(t) = u(t - \frac{\pi}{2}) \sin(t)$

using the information on the last page we have

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u(t-\tfrac{\pi}{2})\sin(t)\}$$
$$= e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin(t+\tfrac{\pi}{2})\}$$
$$= e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin(t)\underbrace{\cancel{\cos(\tfrac{\pi}{2})}}_{0} + \cos(t)\underbrace{\cancel{\sin(\tfrac{\pi}{2})}}_{1}\}$$
$$= e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos(t)\}$$
$$= e^{-\frac{\pi}{2}s}\frac{s}{s^2+1}$$

Some Potentially Useful Information

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s)=\mathcal{L}\{f\}(s)$	$f(t)$	$F(s)=\mathcal{L}\{f\}(s)$
1	$\frac{1}{s},\quad s>0$	e^{at}	$\frac{1}{s-a},\quad s>a$
$t^n,\quad n=1,2,...$	$\frac{n!}{s^{n+1}},\quad s>0$	$\sin(bt)$	$\frac{b}{s^2+b^2},\quad s>0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\quad s>0$	$e^{at}t^n,\quad n=1,2,...$	$\frac{n!}{(s-a)^{n+1}},\quad s>a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\quad s>a$	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\quad s>a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

P1	$\mathcal{L}\{f_1(t)+f_2(t)\}(s)=\mathcal{L}\{f_1(t)\}(s)+\mathcal{L}\{f_2(t)\}(s)$
P2	$\mathcal{L}\{cf(t)\}(s)=c\mathcal{L}\{f(t)\}(s)$
P3	$\mathcal{L}\{e^{at}f(t)\}(s)=\mathcal{L}\{f\}(s-a)$
P4	$\mathcal{L}\{f'\}(s)=s\mathcal{L}\{f\}(s)-f(0)$
P5	$\mathcal{L}\{f''\}(s)=s^2\mathcal{L}\{f\}(s)-sf(0)-f'(0)$
P6	$\mathcal{L}\{f^{(n)}\}(s)=s^n\mathcal{L}\{f\}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-...-f^{(n-1)}(0)$
P7	$\mathcal{L}\{t^nf(t)\}(s)=(-1)^n\frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$

LAPLACE TRANSFORMS OF DISCONTINUOUS FUNCTIONS

Let $F(s)=\mathcal{L}\{f\}(s)$ exist for $s>\alpha\le 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)u(t-a)\}(s)=e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-as}F(s)$ is given by

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a).$$

A corollary of this result is
$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}(s)$