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Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Instructor: Rebecca Tyson Course: MATH 225
Date: Apr 17th, 2024 Time: 3pm Duration: 2 hours 30 minutes.
This exam has 6 questions for a total of 58 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is **not** permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

Question:	1	2	3	4	5	6	Total
Points:	24	4	8	8	6	8	58
Score:							

1. Concept Questions.

[2] (a) Write down a first order ODE that gives rise to the phase line below.

$$\frac{dx}{dt} = (x-a)^2 (b-x)$$

[2] (b) When a first order ODE is exact, the solutions of the ODE represent special curves associated with a three-dimensional function. What are the special curves called and why does the ODE represent these curves?

The special curves are the level curves of the 3-D fn, $F(x,y)$.
The ODE comes from $dF = 0$ (~~so~~, no change in F).

[2] (c) Write down the ODE for a mass-spring oscillator that is critically damped and forced sinusoidally at some arbitrary frequency γ.

$$my'' + 2\sqrt{mk}\,y' + ky = F_0 \cos(\gamma t)$$

[2] (d) In an earthquake, some buildings will collapse while others, right beside the collapsed building, will remain standing. Why?

The buildings that collapse are those with resonance frequency close to the frequency of the seismic waves. Neighbouring buildings can have very different resonance frequencies.

2 (e) Under what circumstances will the solution of a forced undamped mass-spring oscillator give rise to beats? If the forcing frequency is γ, what is the beat frequency?

If the forcing frequency is very close to the resonance frequency ω, we hear beats. The beat frequency is

$$\frac{\gamma - \omega}{2}$$

2 (f) For the underdamped mass-spring oscillator below, write down the **form** of the transient and steady state solutions, and identify each one.

$$\frac{1}{4}y'' + 5y' + 9y = F_0 \cos(\gamma t)$$

$\frac{1}{4}r^2 + 5r + 9 = 0 \iff r^2 + 20r + 36 = 0 \iff r = -10 \pm \sqrt{100-36} = -10 \pm \sqrt{(64)}$
$= -10 \pm 8 = -2, -18$

$$\therefore y(t) = \underbrace{C_1 e^{-2t} + C_2 e^{-18t}}_{\text{transient}} + \underbrace{A\cos(\gamma t) + B\sin(\gamma t)}_{\text{steady-state}}$$

2 (g) Use the formula for R_0 to explain the conditions that give rise to an epidemic for an SIR disease.

$R_0 = \frac{\beta N}{\gamma}$ where β = transmission probability
N = population size
γ = recovery rate

An epidemic will occur if $R_0 > 1$, that is, if the transmission probability or population size are high enough, or if the recovery rate is slow enough.

$\boxed{2}$ (j) When can the method of Laplace Transforms not be used to solve an ODE? Why?

If any of the coefficients or the forcing of the ODE are not of exponential order, then the method of Laplace transforms cannot be used. This is because the Laplace transform is an indefinite integral that is not defined for functions that are not of exponential order.

$\boxed{2}$ (k) List two contexts in which Taylor Series appeared in the course.

- Determining the meaning of $e^{i\theta}$
- Determining the order of the FE and Heun methods
- Deriving the Jacobian.

$\boxed{2}$ (l) The Forward Euler method is *explicit*, while the Backward Euler method is *implicit*. Why?

FE: $\dfrac{y_{n+1}-y_n}{h} = f(t_n, y_n) \iff y_{n+1} = y_n + hf(t_n, y_n)$

The future point, (t_{n+1}, y_{n+1}), is found only using information at the current point, (t_n, y_n).

BE: $\dfrac{y_{n+1}-y_n}{h} = f(t_{n+1}, y_{n+1})$

The future point is found using information at the current time, t_n, and at the future time, t_{n+1}.

End of Concept Questions

2. Consider the ODE
$$\frac{dy}{dx} = 8x^3e^{-2y}.$$

1 (a) Is the ODE linear or nonlinear? Why?

Nonlinear, because the dependent variable y appears nonlinearly in e^{-2y}.

3 (b) Solve the ODE.

$$\frac{dy}{dx} = 8x^3e^{-2y} \Leftrightarrow \int e^{2y}\,dy = \int 8x^3\,dx$$
$$\Leftrightarrow \frac{1}{2}e^{2y} = \frac{8}{4}x^4 + C$$
$$\Leftrightarrow e^{2y} = 4x^4 + 2C$$
$$\Leftrightarrow 2y = \ln(4x^4 + \tilde{C})$$
$$\Leftrightarrow \boxed{y = \frac{1}{2}\ln(4x^4 + \tilde{C})}$$

3. A 3 kg mass is attached to a spring with spring constant $k = 300$ N/m. At time $t = 0$ the mass is pulled right 1 m, given a leftward velocity of 10 m/s, and released.

$\boxed{3}$ (a) Determine the equation of motion for the mass.

$$3y'' + 300y = 0 \iff y'' + 100y = 0$$

characteristic equ: $r^2 + 100 = 0 \iff r = \pm 10i$

$$\therefore y(t) = c_1 \cos(10t) + c_2 \sin(10t)$$

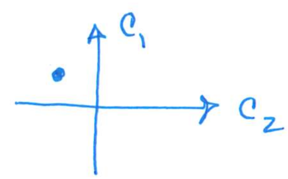
$$\begin{cases} y(0) = 1 \\ y'(0) = -10 \end{cases} \iff \begin{cases} c_1 = 1 \\ 10c_2 = -10 \end{cases} \iff \begin{cases} c_1 = 1 \\ c_2 = -1 \end{cases}$$

$$\therefore \boxed{y(t) = \cos(10t) - \sin(10t)}$$

$\boxed{5}$ (b) What is the maximum rightward displacement and at what time is it achieved?

$y(t) = A\sin(10t + \phi)$ where $A = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

and $\tan(\phi) = \frac{1}{-1} = -1$



$$\therefore \phi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

$$\therefore y(t) = \sqrt{2} \sin\left(10t + \frac{3\pi}{4}\right)$$

The maximum rightward displacement of $\sqrt{2} = y_{max}$ will be first achieved at

$$10t + \frac{3\pi}{4} = \frac{\pi}{2} + n\pi \iff 10t = \frac{2\pi - 3\pi}{4} + n\pi = -\frac{\pi}{4} + n\pi$$

Choose $n = 1$, then $\boxed{t = \frac{3\pi}{40} \text{ s.}}$

[8] 4. Use the method of variation of parameters to find a general solution to the differential equation

$$y'' + 2y' + y = e^{-3t}. \quad (1)$$

Be sure to work from the system of two constraints on $v_1'(t)$ and $v_2'(t)$.

Homogeneous Solution

$r^2 + 2r + 1 = 0 \iff (r+1)^2 = 0 \iff r = -1$

$\therefore\ y_h(t) = c_1 e^{-t} + c_2 t e^{-t}$

Particular Solution

$y_p(t) = v_1(t) e^{-t} + v_2(t) t e^{-t}$

$$\therefore \begin{cases} v_1' e^{-t} + v_2' t e^{-t} = 0 \\ -v_1' e^{-t} + v_2' (e^{-t} - t e^{-t}) = e^{-3t} \end{cases} \iff \begin{cases} v_1' + v_2' t = 0 \\ -v_1' + v_2'(1-t) = e^{-2t} \end{cases} \iff \text{①}$$

$$\text{①} \iff \begin{cases} v_2' = e^{-2t} \\ v_1' = -t e^{-2t} \end{cases} \iff \begin{cases} v_2 = -\frac{1}{2} e^{-2t} \\ v_1 = -\int t e^{-2t}\, dt \end{cases}$$

use integration by parts:

$u = t \quad du = dt$

$dv = e^{-2t} dt \quad v = -\frac{1}{2} e^{-2t}$

$$v_1 = -\left[-\frac{t}{2} e^{-2t} + \frac{1}{2} \int e^{-2t} dt \right] = \frac{t}{2} e^{-2t} + \frac{1}{4} e^{-2t}$$

$$\therefore\ y_p(t) = \left(\frac{t}{2} + \frac{1}{4}\right) e^{-2t} e^{-t} - \frac{1}{2} e^{-2t} t e^{-t} = \frac{1}{4} e^{-3t}$$

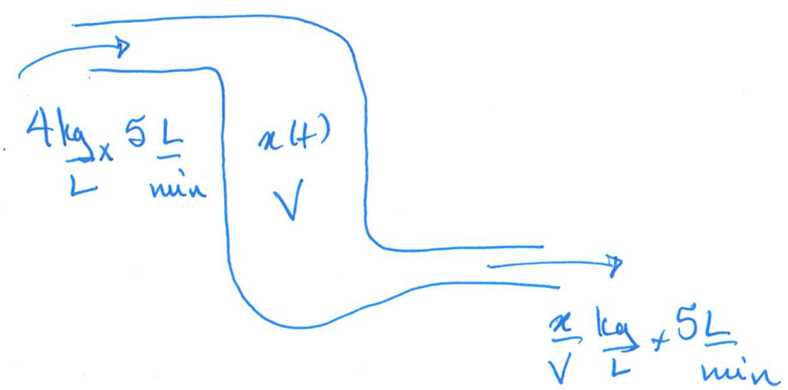
General Solution

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{4} e^{-3t}}$$

$\boxed{6}$ 5. Suppose water containing salt at a concentration of 4 kg/L enters a tank of volume V that initially contains $x(0) = 0$ kg of salt. The rate of flow of brine into the tank is 5 L/min. The outflow rate is the same.

If the mass of salt in the tank after 10 min is 156 kg, show that the volume of the tank is determined by the transcendental equation

$$e^{\frac{-50}{V}} = \frac{V - 39}{V}. \tag{2}$$



$$\frac{dx}{dt} = 4 \cdot 5 - \frac{x}{V} \cdot 5 = 20 - \frac{5}{V}x \iff (1)$$

$$(1) \iff \frac{dx}{dt} + \frac{5x}{V} = 20$$

Integrating factor: $\mu(t) = e^{\int \frac{5}{V} dt} = e^{\frac{5}{V}t}$

Multiplying & integrating we obtain

$$\frac{d}{dt}\left(e^{\frac{5}{V}t}x\right) = 20e^{\frac{5}{V}t} \iff e^{\frac{5}{V}t}x = 20\frac{V}{5}e^{\frac{5}{V}t} + C \iff (11)$$

$$(11) \iff x(t) = 4V + Ce^{-\frac{5}{V}t}$$

ICs: $x(0) = 0 \iff C = -4V \quad \therefore \quad x(t) = 4V(1 - e^{-\frac{5}{V}t})$

We are given

$$x(10) = 156 \iff 156 = 4V(1 - e^{-\frac{50}{V}}) \iff \frac{156}{4V} = 1 - e^{-\frac{50}{V}}$$

$$\iff \boxed{e^{-\frac{50}{V}} = 1 - \frac{39}{V} = \frac{V-39}{V}}$$

as required.

$\boxed{8}$ 6. Solve the IVP below using the method of Laplace Transforms. When taking the Laplace transform of the ODE, identify each of the properties that you use (consult the formula sheet).

$$ty'' - ty' + y = 2, \qquad y(0) = 2, \quad y'(0) = -1$$

Hint: You should get the following ODE for $Y(s)$:

$$\frac{d}{ds}Y(s) + \frac{2}{s}Y(s) = \frac{2}{s^2}$$

$$\mathcal{L}\{ty'' - ty' + y\} = \mathcal{L}\{2\} \Leftrightarrow \text{T.V}$$

$$\text{T.V} \Leftrightarrow \underbrace{\mathcal{L}\{ty''\} - \mathcal{L}\{ty'\} + \mathcal{L}\{y\}}_{\text{P1 and P2}} = \frac{2}{s}$$

$$\Leftrightarrow \underbrace{-\frac{d}{ds}\left[\mathcal{L}\{y''\}\right] + \frac{d}{ds}\left[\mathcal{L}\{y'\}\right]}_{\text{P7}} + Y(s) = \frac{2}{s}$$

$$\Leftrightarrow -\frac{d}{ds}\left[\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\text{P5}}\right] + \frac{d}{ds}\left[\underbrace{sY(s) - y(0)}_{\text{P4}}\right] + Y(s) = \frac{2}{s}$$

$$\Leftrightarrow -2sY(s) - s^2Y'(s) + 2 + Y(s) + sY'(s) + Y(s) = \frac{2}{s}$$

$$\Leftrightarrow s(1-s)Y'(s) + 2(1-s)Y(s) = \frac{2}{s} - 2$$

$$\Leftrightarrow Y'(s) + \frac{2}{s}Y(s) = \frac{2}{s^2}$$

Integrating factor

$$\mu(s) = e^{\int \frac{2}{s}ds} = e^{2\ln(s)} = e^{\ln(s^2)} = s^2$$

Extra workspace for question #6

Multiplying + integrating we obtain

$$s^2 Y'(s) + 2sY(s) = 2 \iff \frac{d}{ds}\left[s^2 Y(s)\right] = 2 \iff \forall,$$

$$\forall, \iff s^2 Y(s) = 2s + C \iff Y(s) = \frac{2}{s} + \frac{C}{s^2}$$

Taking the inverse Laplace Transform we obtain

$$y(t) = 2 + Ct$$

Now apply the ICs:

$$\begin{cases} y(0) = 2 \\ y'(0) = -1 \end{cases} \iff \begin{cases} 2 = 2 \quad \checkmark \\ C = -1 \end{cases}$$

$$\therefore \boxed{y(t) = 2 - t}$$

Some Potentially Useful Information

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$	e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$	$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$	$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

P1 $\mathcal{L}\{f_1(t) + f_2(t)\}(s) = \mathcal{L}\{f_1(t)\}(s) + \mathcal{L}\{f_2(t)\}(s)$

P2 $\mathcal{L}\{cf(t)\}(s) = c\mathcal{L}\{f(t)\}(s)$

P3 $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$

P4 $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

P5 $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$

P6 $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$

P7 $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$