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a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: April 17th, 2023 Time: 3:30pm Duration: ~~3~~ 2 hours 30 minutes.
This exam has 10 questions for a total of 92 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

~~This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.~~

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	6	6	10	8	7	11	10	9	5	20	92
Score:											

1. Consider the first order autonomous ODE

$$\frac{dy}{dt} = y(y-a)(1-y). \tag{1}$$

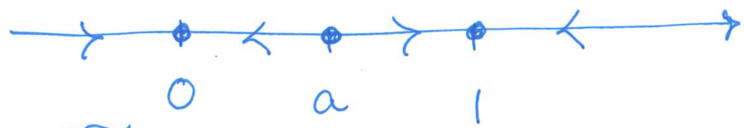
$\boxed{1}$ (a) Show that there are three equilibrium points.

$\frac{dy}{dt} = 0 \Leftrightarrow y = 0,\ y = a,$ or $y = 1$

These are the 3 eq'm pts.

$\boxed{3}$ (b) Sketch the phase line (including stability information) for the cases $0 < a < 1$ and $a > 1$. Explain how you decide the direction of the arrows.

arrow direction flips between eq'm pts because ea one is a monomial



For $y<0$ we know that $y-a<0$ and $1-y>0$, so

$\mathrm{sgn}(y(y-a)(1-y)) = (+ve)(-ve)(+ve) = +ve$

$\boxed{2}$ (c) Suppose that (1) models a population of squeets $S(t)$ that you are studying. Suppose also that currently $0 < a < 1$, but that climate change will increase a past 1 within the next half century. How do you therefore predict that climate change will affect the squeet population?

Currently $S(t) = 1$ ($\because$ it isn't 0 and 1 is a stable eq'm). As a increases through 1, that eq'm point becomes unstable. Depending on the magnitude & direction of stochastic perturbations, the result could be $S(t) \to 0$ or $S(t) \to a > 1$.

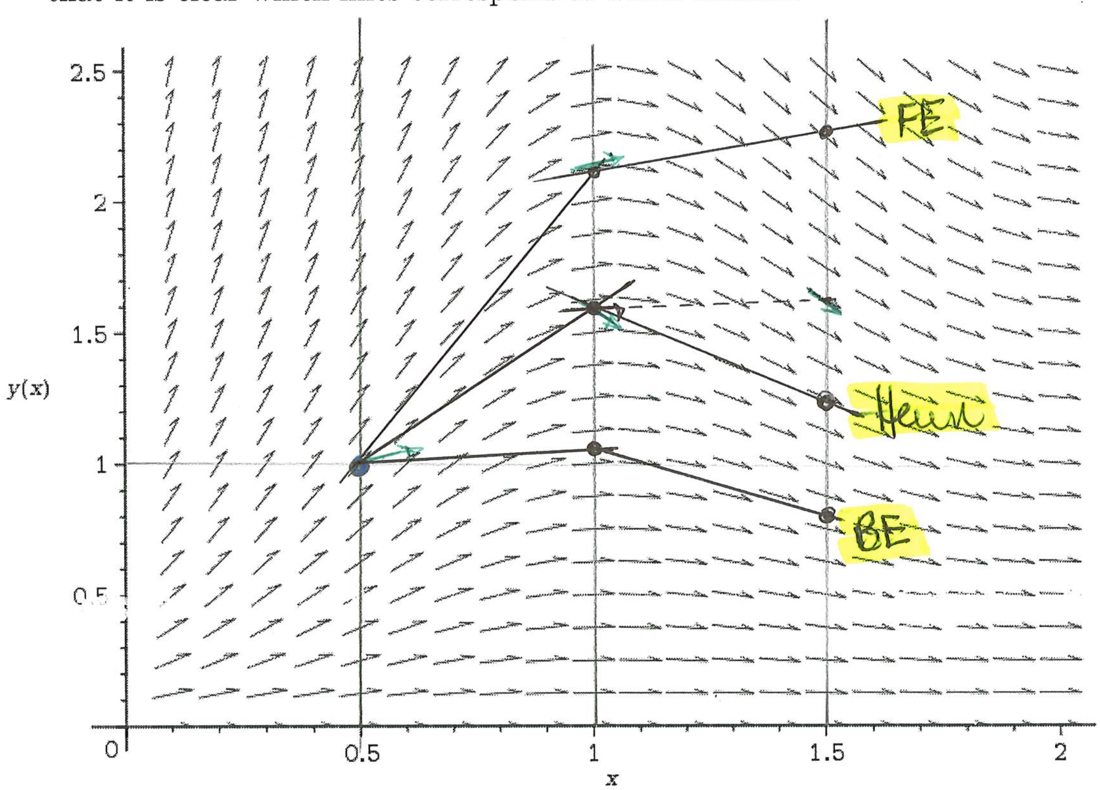
$\boxed{3}$ 2. (a) Given $y' = f(x, y)$, the value of y_{n+1} can be numerically predicted given a starting point (x_n, y_n). Briefly explain how this is done for the Forward Euler (FE), Backward Euler (BE), and Heun methods by completing the sentences below.

- FE: y_{n+1} is found using the slope at the beginning of the timestep
- BE: y_{n+1} is found using the slope at the end of the timestep
- Heun: y_{n+1} is found using the slopes at the beginning and end of the timestep

$\boxed{3}$ (b) Consider the initial value problem

$$\frac{dy}{dx} = \frac{y}{x}\sin(3x), \quad y(0.5) = 1. \tag{2}$$

Using $h = 0.5$ and the direction field below, carefully plot two steps of the solution to (2) using the Backward Euler, Forward Euler, and Heun methods. Make sure that it is clear which lines correspond to which method!



$\boxed{2}$ 3. (a) Determine the most general function $M(\cancel{x,y})$ t,x for which the equation below is exact.

$$M(t,x)dt + (e^t - x)dx = 0$$

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial t}(e^t - x) \iff \frac{\partial M}{\partial x} = e^t \Rightarrow M(t,x) = xe^t + h(t)$$

where $h(t)$ is an arbitrary function of t.

$\boxed{8}$ (b) Using the information above, solve the IVP below.

$$(xe^t + t)dt + (e^t - x)dx = 0, \qquad x(0) = 1 \tag{3}$$

We see that (3) satisfies the conditions to be exact, with $h(t) = t$, so we can be sure that (3) is exact. So there exists

$$F(t,x) = \int (xe^t + t)\,dt = xe^t + \frac{t^2}{2} + g(x) \quad \ldots \quad (3.1)$$

where $g(x)$ is an arbitrary function of x. We know that

$$\frac{\partial F}{\partial x} = e^t - x \iff \frac{\partial}{\partial x}\left[xe^t + \frac{t^2}{2} + g(x)\right] = e^t - x$$

$$\iff e^t + \frac{dg}{dx} = e^t - x \iff \frac{dg}{dx} = -x \iff g(x) = -\frac{x^2}{2} + C$$

Plugging $g(x)$ into (3.1) we obtain

$$F(t,x) = xe^t + \frac{t^2}{2} - \frac{x^2}{2} + C.$$

The solutions are the level curves of F, that is,

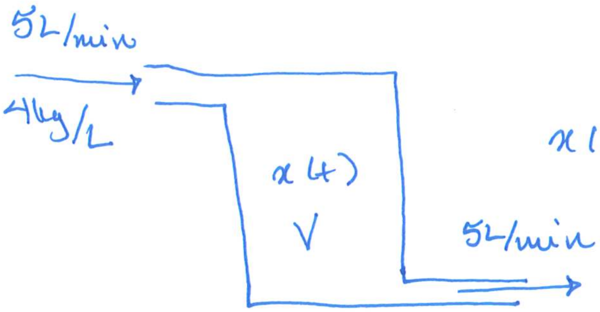
$$xe^t + \frac{t^2}{2} - \frac{x^2}{2} = A.$$

Now apply the IC: $1e^0 + \frac{0^2}{2} - \frac{1^2}{2} = A \iff A = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore\ xe^t + \frac{t^2}{2} - \frac{x^2}{2} = \frac{1}{2}$ is the solution.

$\boxed{8}$ 4. Suppose water containing salt at a concentration of 4 kg/L enters a tank of volume V that initially contains $x(0) = 0$ kg of salt. The rate of flow of brine into the tank is 5 L/min. The outflow rate is the same. If the mass of salt in the tank after 10 min is 156 kg, show that the volume of the tank is determined by the transcendental equation

$$e^{-50/V} = \frac{V-39}{V}. \quad \ldots \quad (4.1)$$



$x(0) = 0$, $x(10) = 156$

$$\frac{dx}{dt} = 4.5 - \frac{x}{V}5 \iff \frac{dx}{dt} + \frac{5}{V}x = 20 \quad \ldots \quad (4.2)$$

We seek an integrating factor:

$$\mu(t) = e^{\int \frac{5}{V}dt} = e^{\frac{5}{V}t}$$

Multiplying (4.2) by $\mu(t)$ we obtain

$$\frac{d}{dt}\left(e^{\frac{5}{V}t}x\right) = 20e^{\frac{5}{V}t} \iff e^{\frac{5}{V}t}x = 20\frac{V}{5}e^{\frac{5}{V}t} + C \iff /1,$$

$$/1, \iff x(t) = 4V + Ce^{-\frac{5}{V}t}$$

Now apply the conditions at $x(0)$ & $x(10)$:

$$x(0) = 0 \iff 4V + C = 0 \iff C = -4V$$

$$x(10) = 156 \iff 156 = 4V\left(1 - e^{-\frac{5}{V}\cdot 10}\right) \iff \frac{39}{V} = 1 - e^{-\frac{50}{V}}$$

$$\iff e^{-\frac{50}{V}} = 1 - \frac{39}{V} \iff e^{-\frac{50t}{V}} = \frac{V-39}{V}$$

as required.

7 5. Consider the unforced mass-spring system whose motion is described by

$$x(t) = \frac{1}{4}e^{-2t}\left(\cos(3t) - \sin(3t)\right). \tag{4}$$

Suppose that $m = 1$. Determine the equation of motion (the ODE) for this mass-spring system, as well as the initial conditions at $t = 0$. All numerical values should be given as fractions, not decimals.

We know that the ODE has the form $mx'' + bx' + kx = 0$. $\because m = 1$ we have $x'' + bx' + kx = 0$. The characteristic equation is

$$r^2 + br + k = 0 \iff r = \frac{-b \pm \sqrt{b^2 - 4k}}{2} = -\frac{b}{2} \pm \frac{\sqrt{b^2-4k}}{2}$$

$\because$ (4) is oscillatory, we know that $r = \alpha \pm i\beta$ where

$$\begin{cases} \alpha = -\frac{b}{2} = -2 \\ \beta = \frac{\sqrt{4k - b^2}}{2} = 3 \end{cases} \iff \begin{cases} b = 4 \\ \sqrt{4k - 16} = 6 \end{cases} \iff \begin{cases} b = 4 \\ 4k - 16 = 36 \end{cases} \iff \begin{cases} b = 4 \\ k = 13 \end{cases}$$

$\therefore$ the ODE is

$$x'' + 4x' + 13x = 0$$

with initial conditions

$$x(0) = \frac{1}{4}e^{-2\cdot 0}\left(\cos(3\cdot 0) - \sin(3\cdot 0)\right) = \frac{1}{4}$$

$$x'(0) = -\frac{2}{4}e^{-2\cdot 0}\left(\cos(3\cdot 0) - \sin(3\cdot 0)\right) + \frac{3}{4}e^{-2\cdot 0}\left(-\sin(3\cdot 0) - \cos(3\cdot 0)\right)$$

$$= -\frac{1}{2} - \frac{3}{4} = -\frac{5}{4}$$

□ 6. Consider the mass-spring system $y'' + by' + y = 0$.

$\boxed{3}$ (a) For what value of b is the system critically damped? Find the general solution in this case.

$r^2 + br + 1 = 0 \Leftrightarrow r = \frac{-b \pm \sqrt{b^2-4}}{2}$

critically damped: $b^2 - 4 = 0 \Leftrightarrow b = 2$

$\therefore r = -\frac{2}{2} = -1$

Solution:
$y(t) = c_1 e^{-t} + c_2 t e^{-t}$

$\boxed{4}$ (b) Suppose now that the critically damped system is subject to forcing satisfying $f(t) = t^{-1}e^{-t}$. Show that the particular solution is given by $y_p(t) = t(\ln(t) - 1)e^{-t}$.

Let $y_p(t) = v_1(t)e^{-t} + v_2(t)\, t e^{-t}$.

Then $\begin{cases} v_1' e^{-t} + v_2' t e^{-t} = 0 \\ -v_1' e^{-t} + v_2'(1-t)e^{-t} = \frac{e^{-t}}{t} \end{cases} \Leftrightarrow \begin{cases} v_1' + t v_2' = 0 \\ -v_1' + (1-t)v_2' = \frac{1}{t} \end{cases} \Leftrightarrow \begin{cases} v_1' = -1 \\ v_2' = \frac{1}{t} \end{cases} \Leftrightarrow \begin{cases} v_1 = -t \\ v_2 = \ln(t) \end{cases}$

$\therefore y_p(t) = -te^{-t} + t\ln(t)e^{-t} = t(\ln(t) - 1)e^{-t}$ as required.

$\boxed{4}$ (c) With initial conditions $y(1) = 2e^{-1}$, $y'(1) = 5e^{-1}$, the system has one maximum at $t = t^*$ Show that t^* is given by the solution of $11 - 6t^* + (1 - t^*)\ln(t^*) = 0$.

Apply the ICs:

$y(t) = c_1 e^{-t} + c_2 t e^{-t} + t(\ln(t) - 1)e^{-t}$; $y'(t) = -c_1 e^{-t} + c_2(1-t)e^{-t} + [\ln(t) - 1 + 1 - t\ln(t) + t]e^{-t}$

$\begin{cases} y(1) = 2e^{-1} \\ y'(1) = 5e^{-1} \end{cases} \Leftrightarrow \begin{cases} c_1 e^{-1} + c_2 e^{-1} - e^{-1} = 2e^{-1} \\ -c_1 e^{-1} + e^{-1} = 5e^{-1} \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 - 1 = 2 \\ -c_1 + 1 = 5 \end{cases} \Leftrightarrow \begin{cases} c_2 = -c_1 + 3 = 7 \\ c_1 = -4 \end{cases}$

At the maximum, we have

$y'(t^*) = 0 \Leftrightarrow +4e^{-t^*} + 7(1-t^*)e^{-t^*} + [\ln(t^*)(1-t^*) + t^*]e^{-t^*} = 0$

$\Leftrightarrow +4 + 7 - 7t^* + \ln(t^*)(1-t^*) + t^* = 0 \Leftrightarrow +11 - 6t^* + \ln(t^*)(1-t^*) = 0$

as required

7. Consider the mass-spring system given by

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = \cos(2t),$$

with fundamental solution set $\{e^{-t/2}\cos((\sqrt{3}/2)t), e^{-t/2}\overset{\sin}{\cancel{\cos}}((\sqrt{3}/2)t)\}$

[7] (a) Derive the steady-state behaviour using the Method of Undetermined Coefficients, and write the behaviour as a single phase-shifted sine.

Let $y_p(t) = A\cos(2t) + B\sin(2t)$

$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$; $y_p'' = -4A\cos(2t) - 4B\sin(2t)$

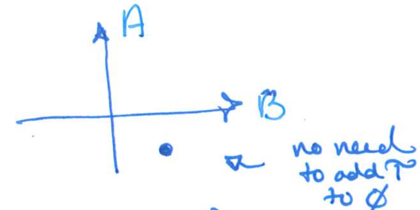
Plugging these into the ODE:

$y_p'' + y_p' + y_p = \cos(2t) \Leftrightarrow (-4A+2B+A)\cos(2t) + (-4B-2A+B)\sin(2t) = \cos(2t)$

$\therefore$ by linear independence

$$\begin{cases} -4A+2B+A = 1 \\ -4B-2A+B = 0 \end{cases} \Leftrightarrow \begin{cases} -3A+2B = 1 \\ -2A-3B = 0 \end{cases} \Leftrightarrow \begin{cases} -2A-3B = 0 \\ -5A-B = 1 \end{cases} \Leftrightarrow \begin{cases} -2A+15A+3 = 0 \\ B = -5A-1 \end{cases} \Leftrightarrow \begin{cases} 13A = -3 \\ B = -5A-1 \end{cases}$$

$$\Leftrightarrow \begin{cases} A = -\frac{3}{13} \\ B = \frac{15}{13} - \frac{13}{13} = \frac{2}{13} \end{cases}$$



$\therefore y_p(t) = -\frac{3}{13}\cos(2t) + \frac{2}{13}\sin(2t) = \mathcal{A}\sin(2t+\phi)$

where $\mathcal{A} = \sqrt{\left(\frac{3}{13}\right)^2 + \left(\frac{2}{13}\right)^2} = \frac{\sqrt{9+4}}{13} = \frac{\sqrt{13}}{13}$; $\phi = \tan^{-1}\left(\frac{-3}{2}\right)$

$\therefore y_p(t) = \frac{\sqrt{13}}{13}\sin\left(2t + \tan^{-1}\left(\frac{-3}{2}\right)\right)$

[3] (b) What is the gain factor for this system? Could it be increased? How? Be specific.

The gain factor is $\sqrt{13}/13$. To increase the gain factor, the forcing frequency should be set to the resonant frequency: $\gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} = \sqrt{\frac{1}{1} - \frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

8. Consider an SIR-type disease where vaccination can provide permanent immunity. If a proportion p of the susceptible population is vaccinated, then the SIR model becomes

$$\frac{dS}{dt} = -\beta(1-p)SI \tag{5a}$$

$$\frac{dI}{dt} = \beta(1-p)SI - \gamma I \tag{5b}$$

$$\frac{dR}{dt} = \gamma I \tag{5c}$$

[3] (a) Draw the compartmental diagram for this disease. What does the parameter β represent?

$$\boxed{S} \xrightarrow{\beta(1-p)I} \boxed{I} \xrightarrow{\gamma} \boxed{R}$$

β is the likelihood of disease transmission upon contact between S & I individuals

[3] (b) What proportion of individuals need to be vaccinated to remove the threat of an epidemic? Assume that $R_0 = 5$ in the absence of vaccination (i.e., if $p = 0$).

$$R_0 = \frac{\beta N}{\gamma} = 5$$

For this model, we have $\beta(1-p)$ instead of just β. So we have the basic reproductive number for this disease, R_p:

$$R_p = \frac{\beta(1-p)N}{\gamma} = 5(1-p)$$

We require $5(1-p) < 1 \iff 1-p < \frac{1}{5} \iff p > \frac{4}{5} = 80\%$

So, to remove the threat of an epidemic, 80% of the population needs to be vaccinated.

3 (c) What assumption do we make about the population in using the compartmental modelling approach? If this assumption is not met, is the real p needed greater or smaller than the one you calculated? Why?

Assumptions: a) Well-mixed S & I compartments
b) Perfect vaccines (100% effective)

a) Real populations are not so well-mixed, so the actual contact rate is a little lower. This would reduce the p needed.

b) If Vaccination isn't 100% effective, then the proportion effectively vaccinated is less than p, which means that p needs to increase.

5 9. Prove property P2 (from the Brief Table of Laplace Transforms included with this test). Assume that $f(t)$ is continuous on $[0, \infty)$ and of exponential order.

$$\mathcal{L}\{f'\}(s) = \int_0^\infty \frac{df}{dt} e^{-st}\, dt \quad \text{let } u = e^{-st} \quad du = -se^{-st}\, dt$$

$$dv = \frac{df}{dt}\, df \quad v = f$$

$$= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\, dt = I$$

$\because$ $f(t)$ is of exponential order we have $\lim_{t \to \infty} f(t)e^{-st} = 0$, so

$$I = 0 - f(0) + s\,\mathcal{L}\{f\}(s)$$

$\therefore$ $\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0)$ as required.

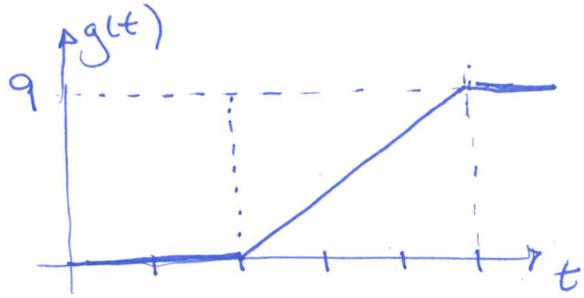
10. Consider the initial value problem

$$y'' + 9y = g(t), \qquad y(0) = y'(0) = 0, \tag{6}$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 2, \\ 3(t-2), & 2 \le t < 5, \\ 9, & t \ge 5. \end{cases} \tag{7}$$

$\boxed{1}$ (a) The function $g(t)$ is known as "ramp loading". Sketch the function.



$\boxed{6}$ (b) Describe the qualitative nature of the solution that you expect from the IVP. Include a sketch. Explain your reasoning.

$\because$ $y(0) = y'(0) = 0$ and the ODE is unforced (RHS $= 0$), we expect the system to be at rest from $t = 0$ to $t = 2$.

At $t \geq 5$, the system has constant forcing, which means that $y_p(t) = A = 1$. At the same time, the LHS of the ODE represents an undamped oscillator with $r = 3$; so $y_h(t) = c_1 \cos(3t) + c_2 \sin(3t)$. So for $t \geq 5$ the behaviour is a periodic oscillation about $y = 1$.

For $2 \leq t < 5$, we expect a smooth transition between the two solutions described above.

$\boxed{2}$ (c) Show that $g(t)$ can be expressed as

$$g(t) = 3(t-2)H(t-2) - 3(t-5)H(t-5). \qquad (8)$$

$$g(t) = 0 + W_{2,5}(t)\,3(t-2) + 9H(t-5)$$
$$= 3(t-2)[H(t-2) - H(t-5)] + 9H(t-5)$$
$$= 3(t-2)H(t-2) + (-3t+6+9)H(t-5) = 3(t-2)H(t-2) - 3(t-5)H(t-5)$$

$\boxed{8}$ (d) Solve the IVP (6) using the method of Laplace Transforms. Within your calculations, identify any properties you use by label (letter name (P#) or equation number - see the "Potentially Useful Information" provided with this test).

$$\mathcal{L}\{y''+9y\} = \mathcal{L}\{g(t)\} \Leftrightarrow 1$$

$$1 \Leftrightarrow \underbrace{s^2Y(s) - sy(0) - y'(0)}_{\text{by P3}} + 9Y(s) = 3\mathcal{L}\{(t-2)H(t-2)\} - 3\mathcal{L}\{(t-5)H(t-5)\}$$

$$\Leftrightarrow (s^2+9)Y(s) = \underbrace{3e^{-2s}\frac{1}{s^2}}_{\text{by (9)}} - \underbrace{3e^{-5s}\frac{1}{s^2}}_{\text{by (9)}}$$

$$\Leftrightarrow Y(s) = 3\frac{1}{s^2}\frac{1}{s^2+9}e^{-2s} - 3\frac{1}{s^2}\frac{1}{s^2+9}e^{-5s}$$

Partial fractions:

$$\frac{1}{s^2(s^2+9)} = \frac{A}{s^2} + \frac{B}{s^2+9} = \frac{(A+B)s^2+9A}{s^2(s^2+9)} \quad \therefore \begin{cases} 9A = 1 \\ A+B = 0 \end{cases} \Leftrightarrow \begin{cases} A = 1/9 \\ B = -1/9 \end{cases}$$

$$\therefore Y(s) = \frac{1}{3}\left[\frac{1}{s^2} - \frac{1}{s^2+9}\right]\left(e^{-2s} - e^{-5s}\right) =$$

$$= \frac{1}{3}\left[(t-2) - \frac{1}{3}\sin(3(t-2))\right]H(t-2) - \frac{1}{3}\left[(t-5) - \frac{1}{3}\sin(3(t-5))\right]H(t-5)$$

(Extra workspace for part d.)

$\boxed{3}$ (e) Write your solution $y(t)$ as piecewise continuous function (i.e., in the format used for (7)), and verify that it matches the behaviour you predicted in part (b).

$$Y(s) = \begin{cases} 0 & 0 \le t < 2 \\ \frac{1}{3}(t-2) - \frac{1}{9}\sin(3(t-2)) & 2 \le t < 5 \\ 1 - \frac{1}{9}\sin(3(t-2)) + \frac{1}{9}\sin(3(t-5)) & 5 \le t \end{cases}$$