a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Feb 6th, 2017 Time: 11:30am Duration: 35 minutes.
This exam has 4 questions for a total of 23 points.

UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

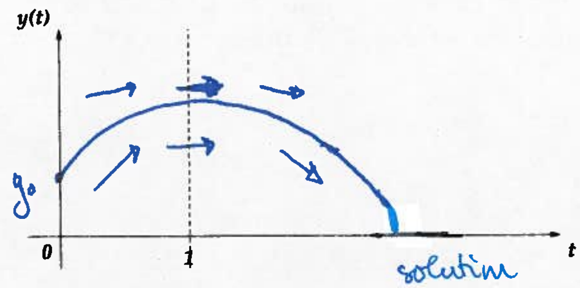
This is a two-stage exam. You have 45 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Question:	1	2	3	4	Total
Points:	7	3	6	7	23
Score:					

1. Consider the initial value problem

$$\frac{dy}{dt} = \frac{2}{y}(1-t), \qquad y(0) = y_0 > 0.$$

[2] (a) On the axes below, sketch a few arrows (about half a dozen) to show the general shape of the direction field.



$\frac{1}{2}$ pt : 0 slope @ $t = 1$

$\frac{1}{2}$ pt : increasing on $0 < t < 1$

$\frac{1}{2}$ pt : decreasing on $1 < t$

$\frac{1}{2}$ pt : shallower slopes as y increases

[5] (b) Solve the initial value problem, and sketch the solution on the direction field above.

$$\frac{dy}{dt} = \frac{2}{y}(1-t) \Leftrightarrow y\,dy = 2(1-t)\,dt$$ 1 pt: separate

$$\Leftrightarrow \frac{y^2}{2} = 2t - t^2 + C$$

$$\Leftrightarrow y^2 = 4t - 2t^2 + \bar{C}$$ 1 pt : implicit soln

$$\Leftrightarrow y = \pm\sqrt{4t - 2t^2 + \bar{C}}$$

Apply the IC:

$y(0) = y_0 \Leftrightarrow y_0^2 = \bar{C}$ and $y_0 > 0$ 1 pt : apply the IC

$$\therefore y = \pm\sqrt{4t - 2t^2 + y_0^2}$$

From the shape of the direction field we know that $y(t) > 0$ initially $\because y_0 > 0$, $\therefore$

$$y = \sqrt{4t - 2t^2 + y_0^2}$$ 1 pt: final solution

1 pt: solution curve (stoppin @ $y = 0$)

$\boxed{3}$ 2. Find the most general function $M(x,y)$ so that the equation below is exact:

$$M(x,y)dx + \left(\sec^2(y) - \frac{x}{y}\right) dy = 0.$$

Test for Exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial x}\left(\sec^2(y) - \frac{x}{y}\right) = -\frac{1}{y}$$

$$\therefore M(x,y) = \int -\frac{1}{y}\, dy = -\ln|y| + h(x)$$

1pt: test for exactness

1pt: integrating to get $M(x,y)$

1pt: constant of integration a function of x.

$\boxed{6}$ 3. Find the general solution to the ODE

$$x\frac{dy}{dx} + 3(y + x^2) = \frac{\sin(x)}{x}.$$

This is a linear ODE. In standard form we have

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin(x)}{x^2} - 3x$$

1pt: standard form

The integrating factor is

$$\mu(x) = e^{\int \frac{3}{x} dx} = e^{3\ln|x|} = e^{\ln|x^3|} = |x^3|$$

2pt: integrating factor

Choose $\mu(x) = x^3$ $\because$ multiplication by -1 has no effect on the solution. Multiplying by $\mu(x)$ the ODE becomes

1pt: choose $\mu(x) = x^3$

$$\frac{d}{dx}(x^3 y) = x\sin(x) - 3x^4 \iff x^3 y = -\frac{3}{5}x^5 + \int x\sin(x)\,dx + C$$

$$\iff x^3 y = -\frac{3}{5}x^5 - x\cos(x) + \sin(x) + C$$

1pt: LHS as perfect product rule

$$\iff y = -\frac{3}{5}x^2 - \frac{\cos(x)}{x^2} + \frac{\sin(x)}{x^3} + \frac{C}{x^3}$$

1pt: final solution

4. Consider the ODE

$$\frac{dy}{dt} = \frac{1}{y-1}\sin\left(\frac{t}{2}\right). \tag{1}$$

$\boxed{2}$ (a) Write the Forward Euler formula for (1) using timesteps of size h.

$$y_{n+1} = y_n + h\left(\frac{1}{y_n - 1}\sin\left(\frac{t_n}{2}\right)\right)$$

1pt: plugging y_n & t_n into slope part

1pt: remainder of the formula

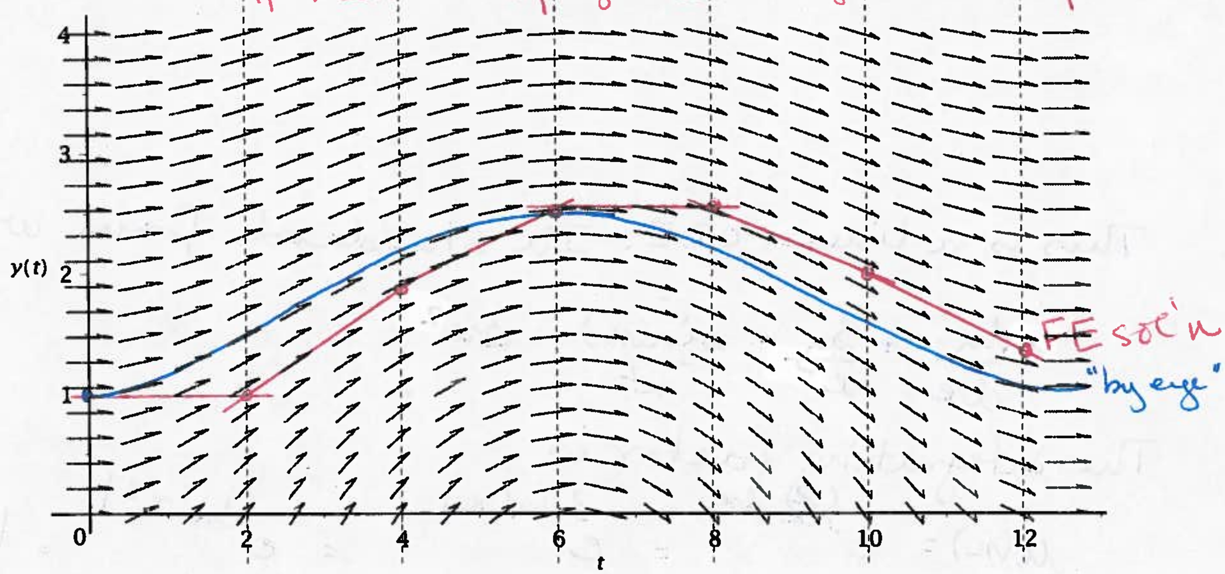
(b) The direction field corresponding to (1) is given below. On that direction field, draw two curves starting at $(t, y) = (0, 1)$:

$\boxed{1}$ i. Sketch the true solution, following the direction field "by eye." 1pt: following arrows

$\boxed{2}$ ii. Using a ruler, carefully draw the Forward Euler solution, using a stepsize of $h = 2$.

1pt: following arrow @ beginning of ea interval

1pt: correct stepsize & correct general shape



$\boxed{2}$ (c) Does the Forward Euler solution underestimate or overestimate the true solution?

The FE solution underestimates the true solution on $0 < t < 6$, & overestimates it on $6 < t < 12$.

1pt: underestimates at first

1pt: overestimates 2nd half