UBC ID #: ________________ NAME (print): ________________________________

Signature: Solutions

INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

a place of mind

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Apr 21st, 2017 Location: EME 050 Time: 1pm Duration 3 hours.
This exam has 8 questions for a total of 74 points.

## SPECIAL INSTRUCTIONS

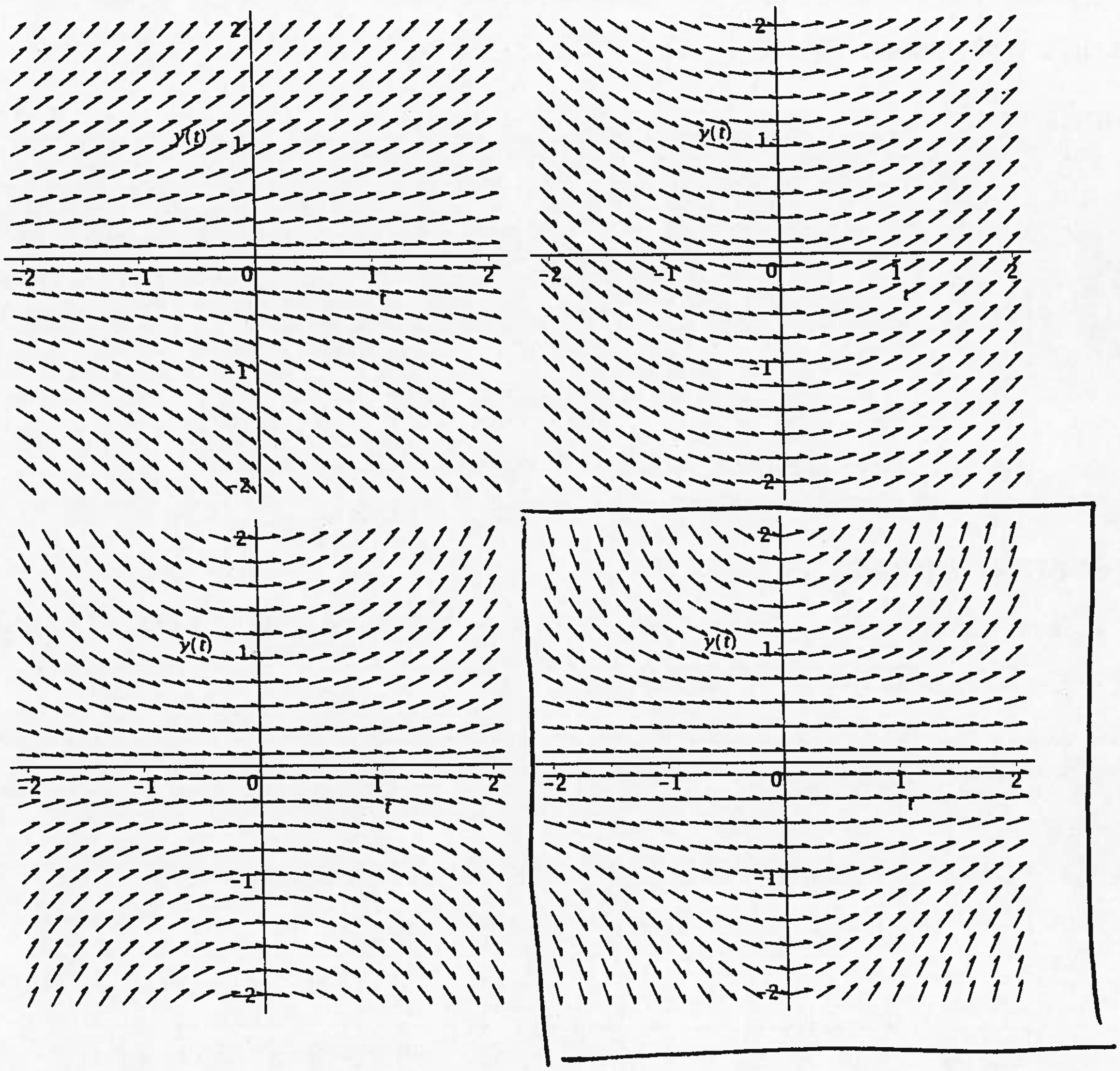
- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points: | 6 | 10 | 4 | 8 | 13 | 9 | 8 | 16 | 74 |
| Score: | | | | | | | | | |

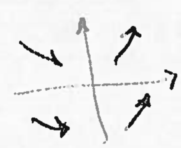
1. Consider the ODE

$$\frac{dy}{dt} = kty^2, \qquad k > 0.$$

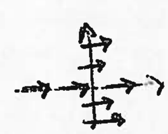
$\boxed{2}$ (a) Which of the direction fields corresponds to the given ODE? Circle it. Then explain your choice. Your arguments must be based on the general behaviour of the direction field and ODE in each quadrant, not simply at particular points.



- RHS of the ODE is $>0 \ \forall t>0$
  $<0 \ \forall t<0$



- $\frac{dy}{dt} = 0$ on $t=0$ and $y=0$



4 (b) Solve the ODE. What method are you using?

Separation of variables

$$\frac{dy}{dt} = kty^2 \Leftrightarrow \int \frac{dy}{y^2} = \int kt\,dt \quad y \neq 0$$

$$\Leftrightarrow -\frac{1}{y} = \frac{kt^2}{2} + \tilde{C}$$

$$\Leftrightarrow y = \frac{-2}{kt^2 + C}$$

$\therefore$ there are two possible solutions:

$$y = 0 \text{ and } y = \underbrace{\frac{-2}{kt^2 + C}}_{\text{family of solutions}}$$

2. A tank, of volume 600 L, is initially filled with $x(0) = 1000$ g of salt. At time $t = 0$, inflow and outflow valves are opened. The concentration of salt in the inflow is given by

$$(2 + \sin(t) \text{ g/L} .$$

The inflow and outflow rates are both 3 L/min. Let $x(t)$ be the amount of salt in the tank at time $t$.

$\boxed{2}$ (a) Write down the ODE that describes the change in the amount of salt in the tank at time $t$.

$$\frac{dx}{dt} = 3(2+\sin(t)) - \frac{3x}{600} \quad \cdots \quad (1)$$

$$x(0) = 1000$$

$\boxed{8}$ (b) Assuming the solution is kept well-stirred, determine the amount of salt in the tank at all times $t > 0$. (Work in **fractions**, not decimals.) *Hint: You may find a useful integral on the formula sheet.*

Putting (1) in standard form we obtain

$$\frac{dx}{dt} + \frac{1}{200}x = 3(2+\sin(t)) \quad \cdots \quad (2)$$

and seek an integrating factor

$$\mu(t) = e^{\int \frac{1}{200} dt} = e^{\frac{1}{200}t}$$

multiplying (2) by $\mu(t)$ we obtain

*Workspace for problem #2.*

$$\frac{d}{dt}\left[e^{\frac{1}{200}t}x\right] = 3(2+\sin(t))e^{\frac{1}{200}t} \quad \Leftrightarrow \quad 1)$$

$$1) \Leftrightarrow e^{\frac{1}{200}t}x = 6\cdot 200\, e^{\frac{1}{200}t} + 3\int e^{\frac{1}{200}t}\sin(t)\,dt$$

$$\Leftrightarrow x = 1200 + \frac{3e^{-\frac{1}{200}t}}{\left(\frac{1}{200}\right)^2+1}\left[\frac{1}{200}\sin(t) - \cos(t)\right]e^{\frac{1}{200}t} + Ce^{-\frac{1}{200}t}$$

$$\Leftrightarrow x = 1200 + \frac{3}{\left(\frac{1}{200}\right)^2+1}\left[\frac{1}{200}\sin(t) - \cos(t)\right] + Ce^{-\frac{1}{200}t}$$

Apply the IC:

$$x(0) = 1000 \Leftrightarrow 1000 = 1200 + \frac{3}{\left(\frac{1}{200}\right)^2+1}(-1) + C$$

$$\Leftrightarrow C = \frac{3}{\left(\frac{1}{200}\right)^2+1} - 200$$

$$\therefore \underline{x(t) = 1200 + \frac{3}{\left(\frac{1}{200}\right)^2+1}}\left[\frac{1}{200}\sin(t) - \cos(t)\right] + \left(\frac{3}{\left(\frac{1}{200}\right)^2+1} - 200\right)e^{-\frac{1}{200}t}$$

3. Consider the initial value problem

$$\frac{du}{dt} = 2u, \qquad u(0) = 1. \tag{1}$$

[1] (a) Write the Backward Euler formula for the given ODE, and solve for $u_{n+1}$.

$$\frac{u_{n+1} - u_n}{h} = 2u_{n+1} \iff u_{n+1} - u_n = 2h u_{n+1}$$

$$\iff u_{n+1} = \frac{u_n}{1-2h}$$

[3] (b) Using the Backward Euler method, and $h = \Delta t = 0.1$, fill in the table below. Show your work!

| $n$ | $u_n$ | $u_{n+1}$ |
|---|---|---|
| 0 | 1 | $\frac{1}{1-2(0.1)} = \frac{1}{1-0.2} = \frac{1}{0.8}$ |
| 1 | $\frac{1}{0.8}$ | $\frac{1}{0.8}\left(\frac{1}{1-0.2}\right) = \frac{1}{(0.8)^2}$ |

4. Consider the ODE

$$t^2y'' - t(t+2)y' + (t+2)y = 0. \qquad (2)$$

One solution of (2) is $y_1(t) = t$.

[7] (a) Use reduction of order to obtain a second linearly independent solution. (Use the full method shown in class - no short cuts!)

Let $y_2(t) = v(t)y_1(t) = tv(t)$ . . . (3)

Then $y_2'(t) = v(t) + tv'(t)$ . . . (4)

$y_2''(t) = 2v'(t) + tv''(t)$

Plugging (3) + (4) into (2) we obtain

$t^2[2v' + tv''] - t(t+2)[v + tv'] + (t+2)vt = 0 \Leftrightarrow$ /1/

/1/ $\Leftrightarrow v''(t^3) + v'(2t^2 - t^3 - 2t^2) = 0$

$\Leftrightarrow v''(t^3) + v'(-t^3) = 0$ . . . (5)

Let $w = v'$. Then (5) becomes

$\frac{dw}{dt} = w \Leftrightarrow \int \frac{dw}{w} = \int dt \Leftrightarrow \ln|w| = t + C \Rightarrow w = Ae^t$

Then $\frac{dv}{dt} = Ae^t \Leftrightarrow v = Ae^t + B$ $\therefore \boxed{y_2(t) = te^t}$

[1] (b) Write the general solution.

$y(t) = C_1 t + C_2 te^t$

5. Consider the mass-spring system governed by

$$\frac{d^2y}{dt^2} + y = 5\sin(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

10 (a) Determine the equation of motion using the Method of Undetermined Coefficients.

Char eqn:

$r^2 + 1 = 0 \Leftrightarrow r = \pm i$

$\therefore y_h(t) = c_1 \cos(t) + c_2 \sin(t)$

particular solution

Let $y_p(t) = At\cos(t) + Bt\sin(t)$

$y_p'(t) = A\cos(t) + B\sin(t) + t(-A\sin(t) + B\cos(t))$

$y_p''(t) = [-A\sin(t) + B\cos(t)]2. + t(-A\cos(t) - B\sin(t))$

plugging $y_p(t)$ into the ODE we obtain

$2[-A\sin(t) + B\cos(t)] - t[A\cos(t) + B\sin(t)] + t[A\cos(t) + B\sin(t)] = 5\sin(t) \Leftrightarrow 1/$

$1/ \Leftrightarrow -A\sin(t) + B\cos(t) = 5\sin(t)$

$\Leftrightarrow A = -5, \; B = 0$

$\therefore y(t) = c_1\cos(t) + c_2\sin(t) - 5t\cos(t)$

*Workspace for problem #5.*

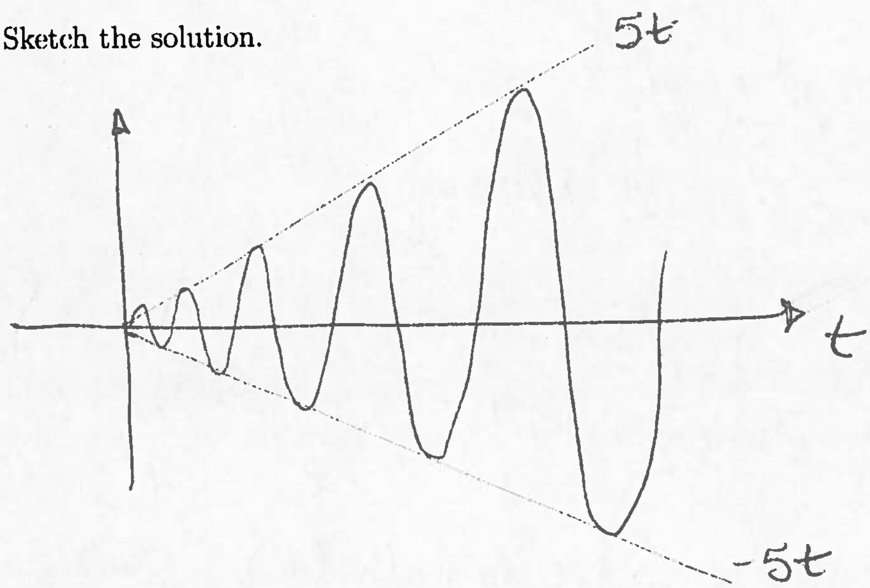
Apply the ICs:

$y(0) = 0 \Leftrightarrow c_1 = 0$

$y'(0) = 1 \Leftrightarrow c_2 - 5 = 1 \Leftrightarrow c_2 = 6$

$\therefore\ y(t) = 6\sin(t) - 5t\cos(t)$

[2] (b) Sketch the solution.




[1] (c) Name the behaviour.

resonance

$\boxed{9}$ 6. Find the general solution of the ODE

$$y'' - 2y' + y = \frac{e^t}{t}, \qquad t > 0.$$

Use fractions, not decimals, in your work and answer. *Use the full method shown in class - no short cuts! Also, make sure that your solution does not contain any redundancies.*

(i) homogeneous sol'n

$r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0 \Leftrightarrow r = 1$

$\therefore y_h(t) = c_1 e^t + c_2 t e^t$

(ii) particular sol'n

$y_p(t) = v_1 e^t + v_2 t e^t$

$\therefore$ $v_1$ & $v_2$ must satisfy

$$\begin{cases} v_1' e^t + v_2' t e^t = 0 \\ v_1' e^t + v_2' (e^t + t e^t) = \dfrac{e^t}{t} \end{cases} \Leftrightarrow \begin{cases} v_2' e^t = \dfrac{e^t}{t} \\ v_1' = -v_2' t \end{cases} \Leftrightarrow ///$$

$$/// \Leftrightarrow \begin{cases} v_2(t) = \ln(t) \quad (t > 0) \\ v_1(t) = -t \end{cases}$$

same as $y_2(t)$)

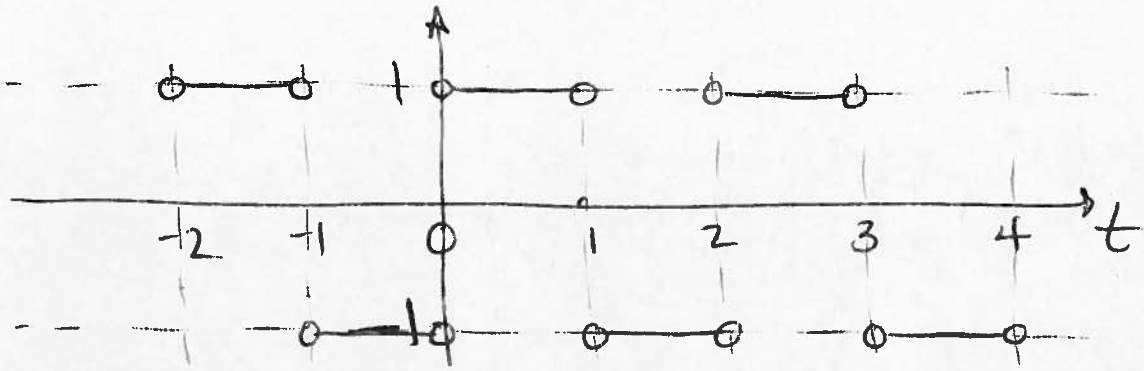
$\therefore y_p(t) = -te^t + t \ln(t) e^t$

$\therefore y(t) = c_1 e^t + c_2 t e^t + t \ln(t) e^t$

7. Consider the periodic function defined by

$$f(t) = \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \end{cases} \text{ and } f(t) \text{ has period 2.} \qquad (3)$$

$\boxed{2}$ (a) Sketch three periods of the function $f(t)$. Include axis labels.




$\boxed{6}$ (b) Find the Laplace transform of $f(t)$.

$$F_T(s) = \int_0^2 f(t)e^{-st}\,dt = \int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_0^1 - \left.\frac{e^{-st}}{-s}\right|_1^2 = -\frac{e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s}$$

$$= \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{(1-e^{-s})^2}{s}$$

$$\therefore F(s) = \frac{1}{1-e^{-2s}} \frac{(1-2e^{-s}+e^{-2s})}{s} = \frac{(1-e^{-s})^2}{s(1-e^{-2s})}$$

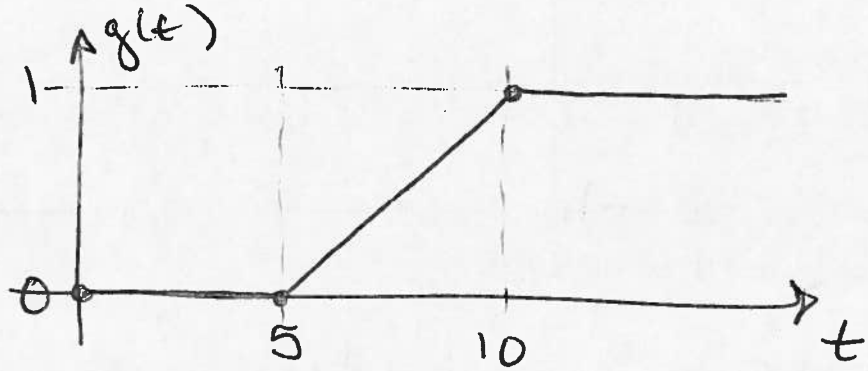
8. Consider the initial value problem

$$y'' + 4y = g(t), \qquad y(0) = y'(0) = 0,$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 5, \\ \frac{t-5}{5}, & 5 \le t < 10, \\ 1, & t \ge 10. \end{cases}$$

[2] (a) The function $g(t)$ is known as "ramp loading". Sketch the function. Include axis labels.




[4] (b) Express $g(t)$ as a sum of Window and Heaviside functions, then as a sum of only Heaviside functions.

$$g(t) = W_{5,10}(t)\,\frac{(t-5)}{5} + H(t-10)$$

$$= \frac{(t-5)}{5}\left[H(t-5) - H(t-10)\right] + H(t-10)$$

$$= \frac{(t-5)}{5} H(t-5) + \left(1 - \frac{t}{5} + 1\right) H(t-10)$$

$$= \frac{(t-5)}{5} H(t-5) - \frac{(t-10)}{5} H(t-10)$$

[10] (c) Solve the IVP using the method of Laplace Transforms.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}\ (1),$$

*Workspace for problem #8.*

$$s^2 Y(s) - s y(0) - y'(0) + 4 Y(s) = \frac{1}{5} e^{-5s} \frac{1}{s^2} - \frac{1}{5} e^{-10s} \frac{1}{s^2}$$

$$\Leftrightarrow Y(s)\,(s^2+4) = \frac{1}{5s^2}\left(e^{-5s} - e^{-10s}\right)$$

$$\Leftrightarrow Y(s) = \frac{1}{5}\left[\frac{1}{s^2} \cdot \frac{1}{s^2+4}\right]\left(e^{-5s} - e^{-10s}\right)$$

partial fractions

$$\frac{1}{s^2} \cdot \frac{1}{s^2+4} = \frac{A}{s^2} + \frac{B}{s^2+4} = \frac{As^2 + 4A + Bs^2}{s^2(s^2+4)}$$

$$\therefore \begin{cases} A + B = 0 \\ 4A = 1 \end{cases} \Leftrightarrow \begin{cases} B = -\frac{1}{4} \\ A = \frac{1}{4} \end{cases}$$

$$\therefore Y(s) = \frac{1}{20}\left[\frac{1}{s^2} - \frac{1}{2}\frac{2}{s^2+4}\right]\left(e^{-5s} - e^{-10s}\right)$$

and so

$$y(t) = \frac{1}{20}\left(\left[(t-5) - \frac{1}{2}\sin(t-5)\right]H(t-5) + \left[(t-10) - \frac{1}{2}\sin(t-10)\right]H(t-10)\right)$$