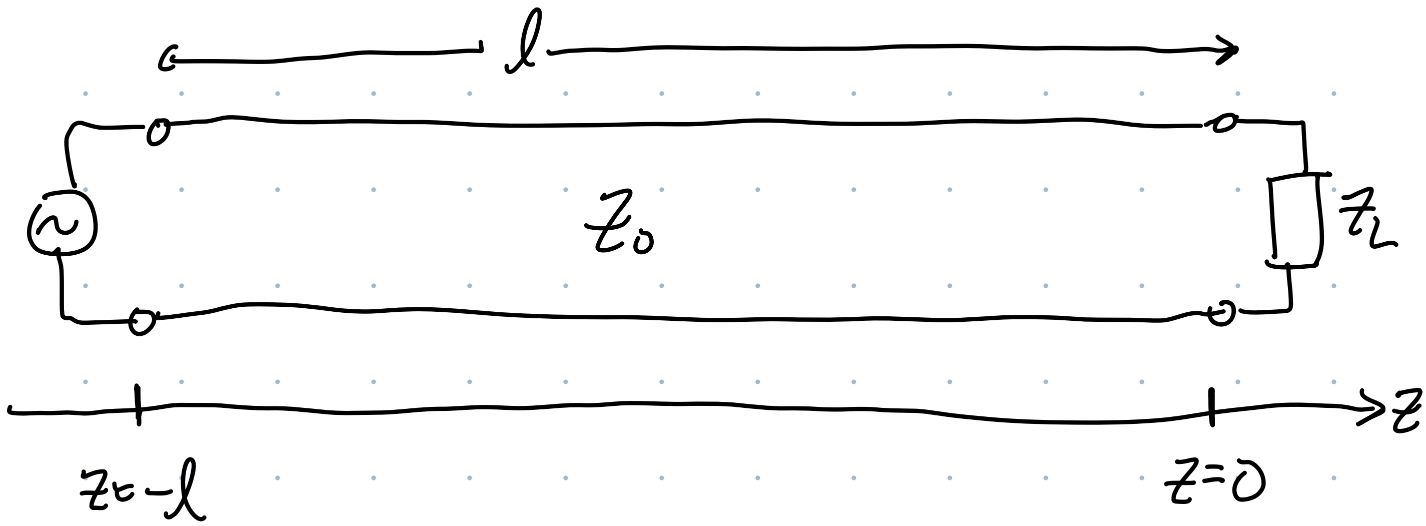
PHYS 331 Oct. 7, 2025

Two weeks ago... **Transmission Lines**



Summary:

$$v(z,t) = V(z)e^{j\omega t} \qquad i(z,t) = I(z)e^{j\omega t}$$

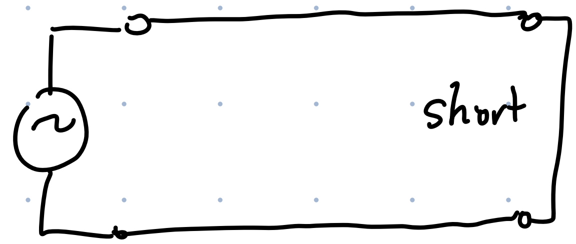
$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z} \qquad I(z) = \frac{1}{Z_0}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$$

$$\beta = \omega\sqrt{L_l C_l} = \frac{2\pi}{\lambda} \qquad s = \frac{1}{\sqrt{L_l C_l}} \qquad Z_0 = \sqrt{\frac{L_l}{C_l}}$$

Reflection Coefficient: $\Gamma = \dfrac{V_-}{V_+} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

Today: Special Cases of Γ

① $Z_L = 0$ (short circuit)

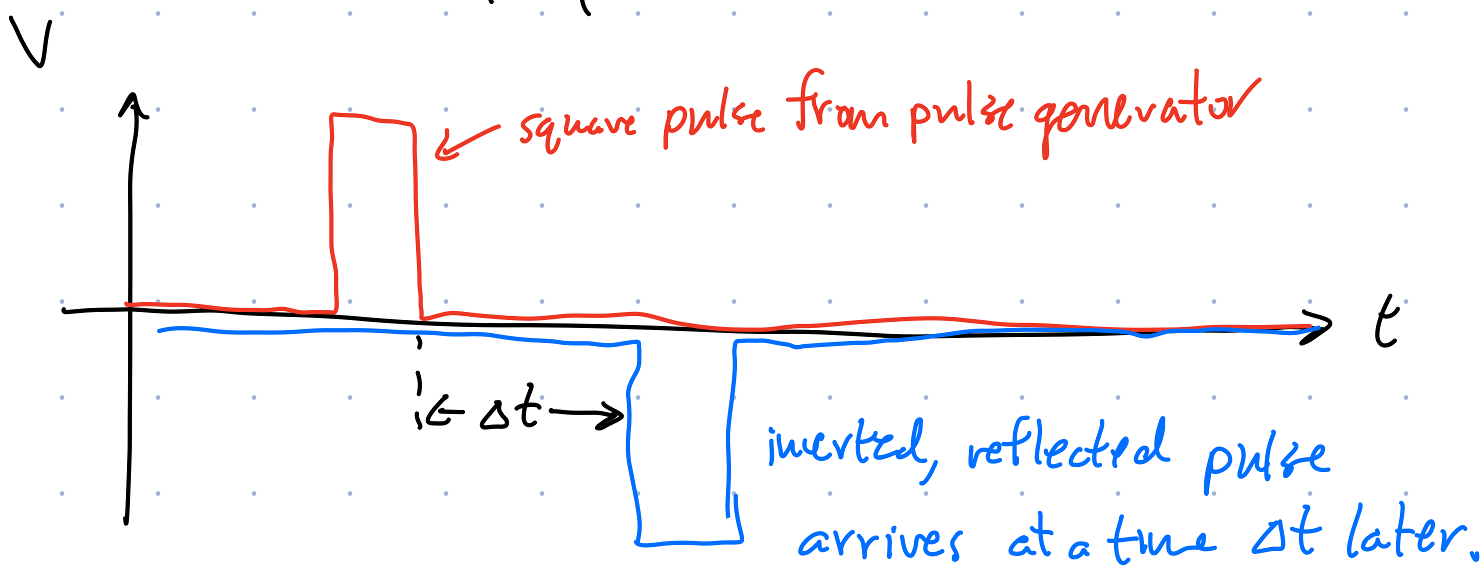


$$\Gamma = \frac{0 - Z_0}{0 + Z_0} = -1$$

reflected wave is inverted.

Corresponds to a perfect 100% reflection

$|\Gamma| = 1$.

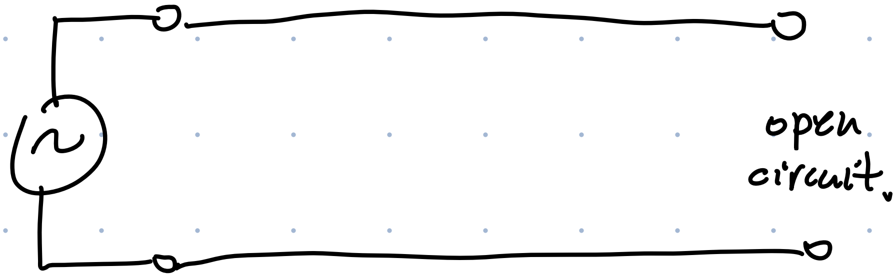


$$\Delta t = \frac{2l}{s}$$

Δt is time for pulse to travel the TL length l, reflect at $Z_L = 0$, and return to input.

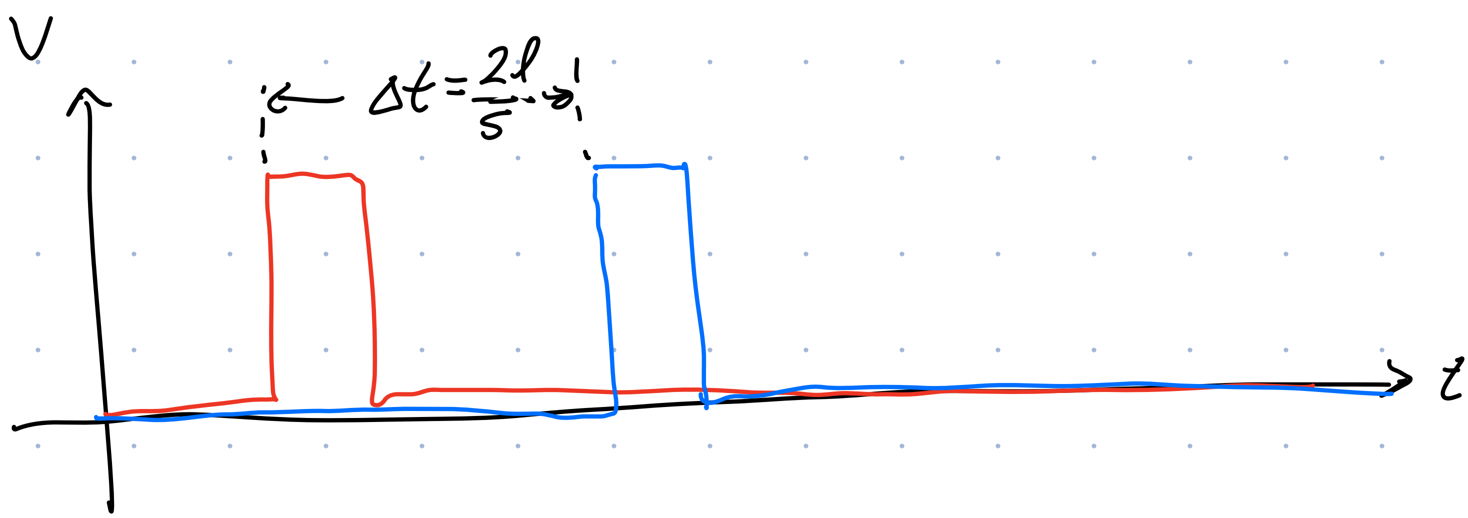
② $Z_L \to \infty$ (open circuit)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad Z_L \gg Z_0$$



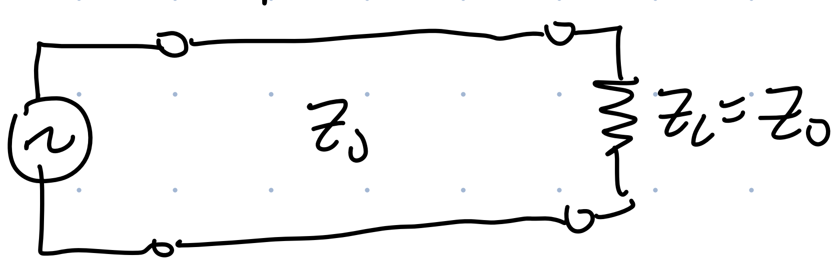
$$\approx \frac{Z_L}{Z_L} = +1$$

Again, get a perfect reflection, but this time w/o an inversion.



③ $Z_L = Z_0$

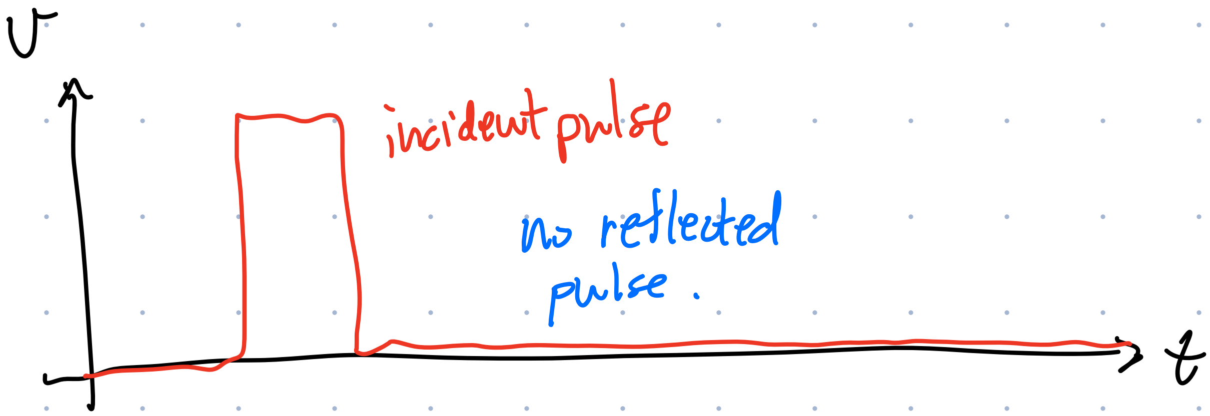
Set the load impedance Z_L to be equal to the characteristic impedance Z_0 of the TL.



Typically $Z_0 = 50\,\Omega$ for coaxial cables used in a physics lab.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0}{2Z_0} = 0$$

No reflection at all.



This is called impedance matching, We usually desire to acheive $Z_L = Z_0$ to efficiently transfer power from source to some device.

When are TL effects important and when can they be ignored?

$$c = \lambda f \qquad \therefore \lambda = \frac{c}{f}$$

take $f = 10\,\text{kHz}$ (typical freq in PHYS 231)

$$\Rightarrow \lambda = \frac{3\times10^8\,\text{m/s}}{10\times10^3\,{}^1/_s} = 3\times10^4\,\text{m} = 30\,\text{km}!$$

Here $\lambda \gg$ length of cable.

take $f = 3\,\text{GHz}$ (microwave regime)

$$\lambda = \frac{3\times10^8\,\text{m/s}}{3\times10^9\,{}^1/_s} = 0.1\,\text{m} = 10\,\text{cm}$$

comparable or shorter than typical cable.

We will find that TL effects are important when the wavelength is comparable or shorter than cable lengths.

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$$ volt. amp. as a fcn of position z.

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{L_\ell C_\ell}$$

The coaxial cables in the lab are called RG-58 and designed s.t.

$$\left.\begin{aligned} C_\ell &\approx 80\,\text{pF/m} \\ L_\ell &\approx 0.20\,\mu\text{H/m} \end{aligned}\right\} \quad Z_0 = \sqrt{\frac{L_\ell}{C_\ell}} \approx 50\,\Omega$$

$\beta = \omega\sqrt{L_\ell C_\ell}$ let's take as a first example
$\omega = 2\pi(10\,\text{kHz})$

$$\Rightarrow \beta = 2.5\times 10^{-4}\,\text{m}^{-1}$$

If we take $z \approx 1\,\text{m}$, $\beta z = 2.5\times 10^{-4} \ll 1$

$$V(z) = V_+ \underbrace{e^{-j\beta z}}_{1} + V_- \underbrace{e^{j\beta z}}_{1} = V_+ + V_- = \text{const.}$$

For low freq, our TL voltage (and current) amplitudes are uniform along the TL length.

If, on the other hand, we take $\omega = 2\pi(3GHz)$

$\beta = \omega\sqrt{L_e C_e} = 75\,m^{-1}$

In this case, for $0 < z < 1\,m$, the product βz is <u>not</u> always small.

Cannot approx $e^{\pm j\beta z}$ as 1.

In fact, $e^{\pm j\beta z} = \cos\beta z \pm j\sin\beta z$

Here, we find that the voltage and current amp. osc. along the length of the TL.

$\rightarrow$ Gives rise to all the TL effects.

Impedance at an arbitrary position z along the TL.

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$$

$$I(z) = \frac{1}{Z_0}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$$

At position z, the impedance is given by:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_+ e^{-j\beta z} + V_- e^{j\beta z}}{V_+ e^{-j\beta z} - V_- e^{j\beta z}}$$

impedance

position.

① replace V_- w/ ΓV_+

$$Z(z) = Z_0 \frac{\cancel{V_+} e^{-j\beta z} + \Gamma \cancel{V_+} e^{j\beta z}}{\cancel{V_+} e^{-j\beta z} - \Gamma \cancel{V_+} e^{j\beta z}}$$

② replace $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{j\beta z}}{e^{-j\beta z} - \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{j\beta z}} \qquad \frac{(Z_L + Z_0)}{(Z_L + Z_0)}$$

③ mult by $1 = \dfrac{Z_L + Z_0}{Z_L + Z_0}$

$$Z(z) = Z_0 \frac{(Z_L + Z_0) e^{-j\beta z} + (Z_L - Z_0) e^{j\beta z}}{(Z_L + Z_0) e^{-j\beta z} - (Z_L - Z_0) e^{j\beta z}}$$

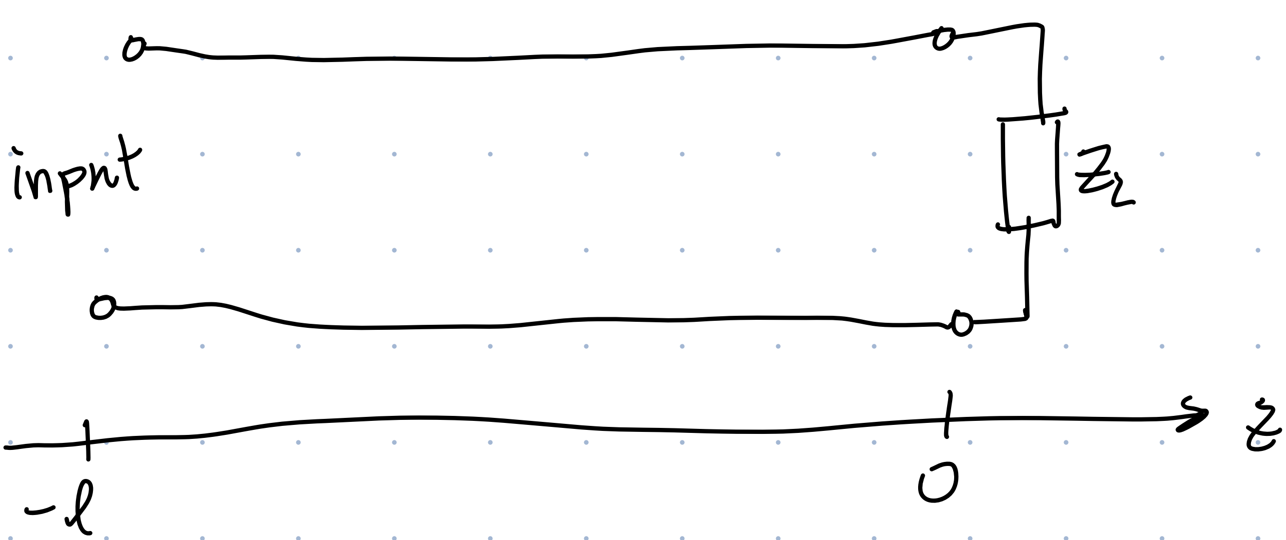
④ Collect Z_L, Z_0 factors

$$Z(z) = Z_0 \frac{Z_L \overbrace{\left(e^{-j\beta z} + e^{j\beta z}\right)}^{\cancel{2}\cos\beta z} - Z_0 \overbrace{\left(e^{j\beta z} - e^{-j\beta z}\right)}^{\cancel{2}j\sin\beta z}}{Z_0 \underbrace{\left(e^{j\beta z} + e^{-j\beta z}\right)}_{\cancel{2}\cos\beta z} - Z_L \underbrace{\left(e^{j\beta z} - e^{-j\beta z}\right)}_{\cancel{2}j\sin\beta z}}$$

⑤ mult by $1 = \dfrac{\frac{1}{\cos\beta z}}{\frac{1}{\cos\beta z}}$

$$\boxed{Z(z) = Z_0 \frac{Z_L - jZ_0 \tan\beta z}{Z_0 - jZ_L \tan\beta z}}$$

Impedance at any position z along the TL terminated by Z_L.



To get the impedance at the TL input, Z_{in}, set $z=-l$ and use $\tan(-x) = -\tan x$

$$\boxed{Z_{in} \equiv Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}}$$

Impedance looking into a TL of length l terminated by impedance Z_L.

Special Cases:

① $\beta l = n\pi$ Select l or ω s.t. we satisfy $\beta l = n\pi$.

$$\frac{2\cancel{\pi}}{\lambda} l = n\cancel{\pi} \Rightarrow l = n\left(\frac{\lambda}{2}\right)$$

Equivalent to setting l to be an integer multiple of have wavelengths.

$$\tan \beta l = \tan(n\pi) = 0$$

$$Z_{in} = \cancel{Z_0} \frac{Z_L}{\cancel{Z_0}} \Rightarrow Z_{in} = Z_L$$

As if the TL is not there. Like we've connected Z_L directly to the source.

(2) $\beta l = n'\left(\frac{\pi}{2}\right)$ where n' is an <u>odd</u> integer.

$$\frac{2\cancel{\pi}}{\lambda} l = n' \frac{\cancel{\pi}}{2} \Rightarrow l = n'\left(\frac{\lambda}{4}\right)$$

length of TL is an odd mult. of quarter wavelengths.

$$\tan \beta l = \tan\left(n' \frac{\pi}{2}\right) \to \pm \infty$$

In this case:

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

$$\approx Z_0 \frac{\cancel{j} Z_0 \cancel{\tan} \cancel{\beta l}}{\cancel{j} Z_L \cancel{\tan} \cancel{\beta l}}$$

$$= \frac{Z_0^2}{Z_L}$$

In effect, we're transforming impedances from Z_L to $\frac{Z_0^2}{Z_L}$.

Eg. If $Z_0 = 50\,\Omega$ & $Z_L = 25\,\Omega$, then $\frac{Z_0^2}{Z_L} = 100\,\Omega$

transformed from a small Z_L to a larger Z_0^2/Z_L

③ $Z_L = Z_0$ (impedance matching condition)

$$Z_{in} = Z_0 \frac{\cancel{\cancelto{Z_0}{Z_L} + jZ_0 \tan\beta l}}{\cancel{Z_0 + j\cancelto{Z_0}{Z_L} \tan\beta l}}$$

If $Z_L = Z_0$,

then $Z_{in} = Z_0$

Usually desired. Get impedance matching at all freq. $\rightarrow$ Efficient power transfer w/ no reflections.