PHYS 331 Sept. 2, 2025

Review of Complex Nos. & their Physical Meaning

Recall that we can express <u>any</u> complex number in two differents:

① Mag. & Phase:

$$z = |z|e^{j\phi} \qquad (j \equiv \sqrt{-1})$$
$$= |z| \underbrace{(\cos\phi + j\sin\phi)}_{\text{Euler's Eq'n } e^{\pm j\phi} = \cos\phi \pm j\sin\phi}$$

② real & imaginary components

$$z = \underbrace{x}_{\text{real component}} + j\underbrace{y}_{\text{imagary compoment.}}$$

Can easily switch between the two representations:

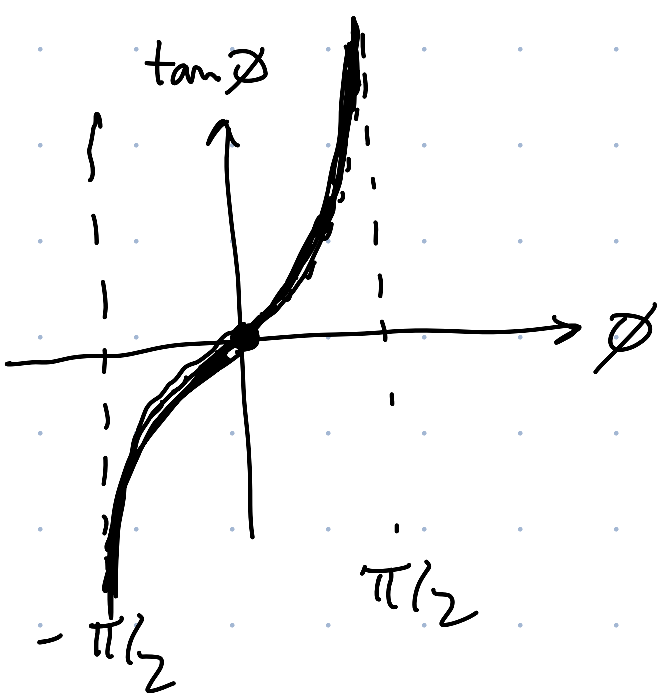
$$x + jy = |z|(\cos\phi + j\sin\phi)$$

Equating real & imaginary components on left & right gives:

$$\boxed{\begin{array}{ll} x = |z|\cos\phi & y = |z|\sin\phi \\ |z| = \sqrt{x^2+y^2} & \frac{y}{x} = \tan\phi \\ & \phi = \tan^{-1}\left(\frac{y}{x}\right) \end{array}}$$

If $\mathrm{Im}[z] = 0$, $y = 0$, $\tan\phi = 0$

$x \neq 0$ $\hookrightarrow \phi = 0$.



A purely real number w/ $y = \mathrm{Im}[z] = 0$ has a phase of zero.

If y is large and $\frac{y}{x} > 0$ (positive).

then $\phi \to +\pi/2$

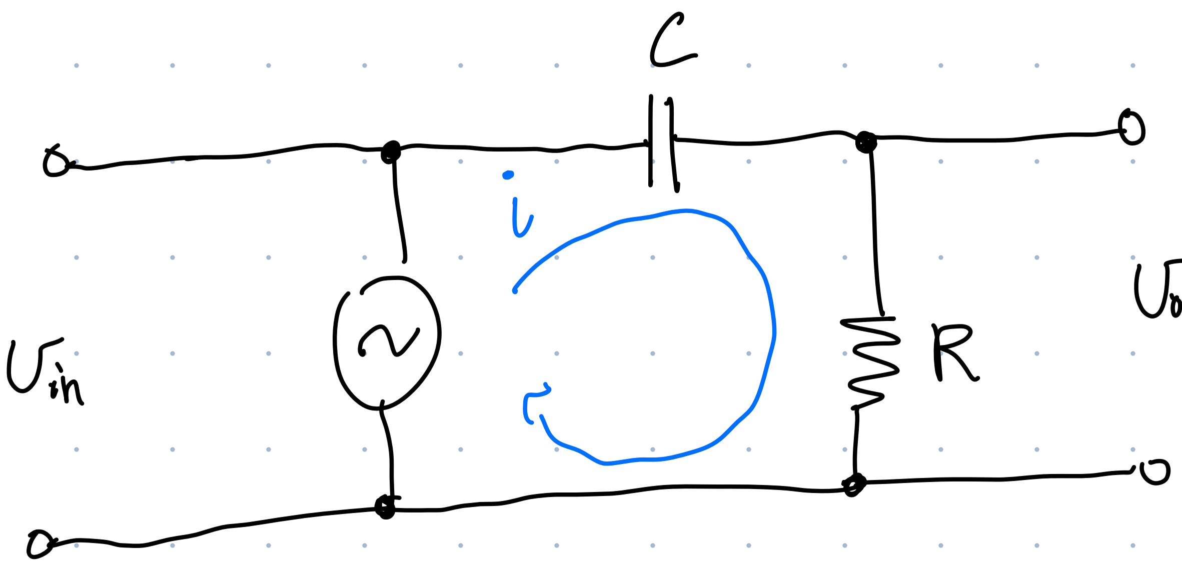
If y is large and $\frac{y}{x} < 0$ (negative)

then $\phi \to -\pi/2$

The relative size and sign of x, y determine the phase ϕ. Real nos, w/ $y=0$, have $\phi = 0$.

Alternatively, a nonzero imaginary component y implies a nonzero phase.

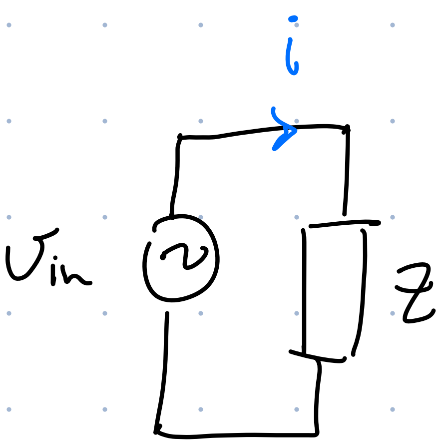
Consider an example from electronics.
RC series circuit.



$v_{out} = iR$

Write $\quad v_{in} = V_{in} e^{j\omega t} \qquad v_{out} = V_{out} e^{j(\omega t + \phi)}$

To fully determine v_{out}, need to find the amplitude V_{out} & phase ϕ.



$$i = \frac{v_{in}}{Z} \quad 1$$

For our RC circuit

$$Z = \frac{1}{j\omega C} + R$$

$$Z = \frac{1 + j\omega RC}{j\omega C}$$

$$\therefore \quad i = v_{in} \left(\frac{j\omega C}{1 + j\omega RC} \right) \frac{(1 - j\omega RC)}{(1 - j\omega RC)}$$

From (1)

$$= v_{in} \left(\frac{j\omega C + \omega^2 RC^2}{1 + (\omega RC)^2} \right)$$

$$i = \omega C v_{in} \left(\frac{j + \omega RC}{1 + (\omega RC)^2} \right)$$

$$\therefore v_{out} = iR = \omega RC \, v_{in} \left(\frac{j + \omega RC}{1 + (\omega RC)^2} \right)$$

Recall that $v_{in} = \boxed{V_{in} e^{j\omega t}}$

$$v_{out} = V_{out} e^{j(\omega t + \phi)} = \boxed{V_{out} e^{j\omega t} e^{j\phi}}$$

$$V_{out}\,\cancel{e^{j\omega t}}\,\underbrace{e^{j\phi}}_{\cos\phi + j\sin\phi \ \text{(Euler)}} = V_{in}\,\cancel{e^{j\omega t}}\,\frac{\omega RC(j+\omega RC)}{1+(\omega RC)^2}$$

$$V_{out}(\cos\phi + j\sin\phi) = V_{in}\,\frac{\omega RC(j+\omega RC)}{1+(\omega RC)^2}$$

real & imaginary parts on left must equal real & imaginary parts on right.

Real:

$$V_{out}\cos\phi = V_{in}\,\frac{(\omega RC)^2}{1+(\omega RC)^2} \equiv X$$

Imaginary:

$$V_{out}\sin\phi = V_{in}\,\frac{\omega RC}{1+(\omega RC)^2} \equiv Y$$

X is the real component of V_{out}

Y is the imaginary component of V_{out}

Our eventual goal is to see how we can use a lock-in detector to directly meas. X & Y.

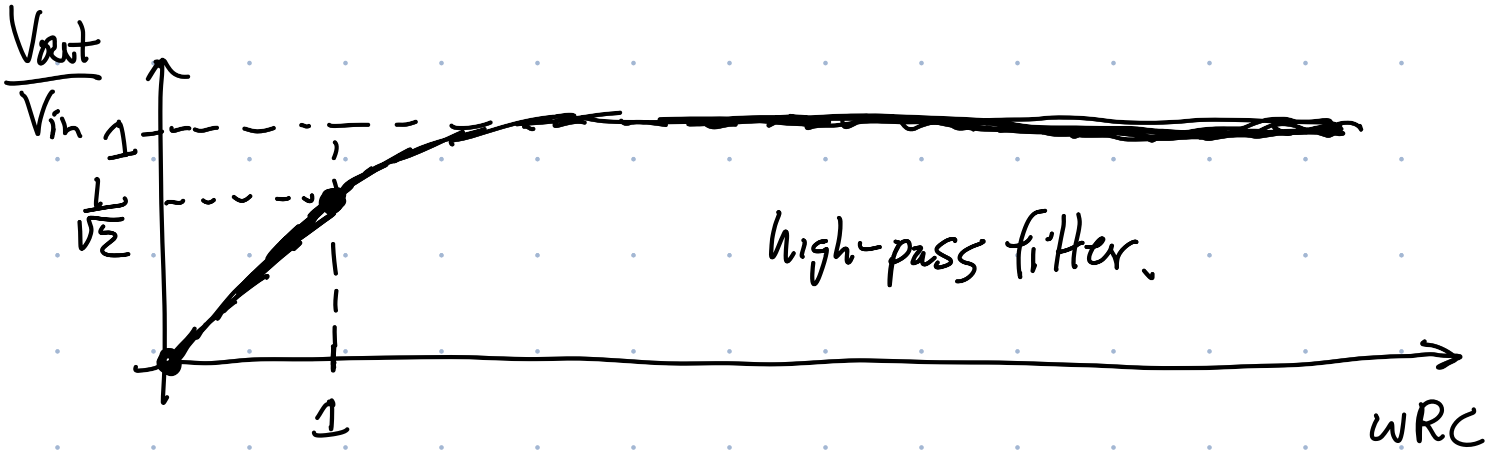
To find the amplitude of V_{out}, evaluate $\sqrt{X^2+Y^2}$

$$V_{out} = \sqrt{X^2+Y^2} = \sqrt{\frac{V_{in}^2(\omega RC)^4}{(1+(\omega RC)^2)^2} + \frac{V_{in}^2(\omega RC)^2}{(1+(\omega RC)^2)^2}}$$

$$= \frac{V_{in}\,\omega RC}{1+(\omega RC)^2}\sqrt{(\omega RC)^2+1}$$

$$\boxed{\therefore V_{out} = V_{in}\frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}$$

Amplitude of V_{out} in terms of known quantities.

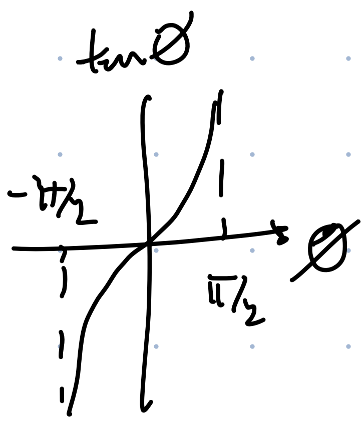


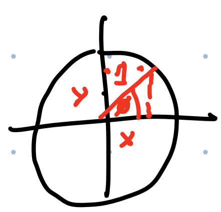
To get the phase, consider $\frac{y}{x} = \tan\varnothing$.

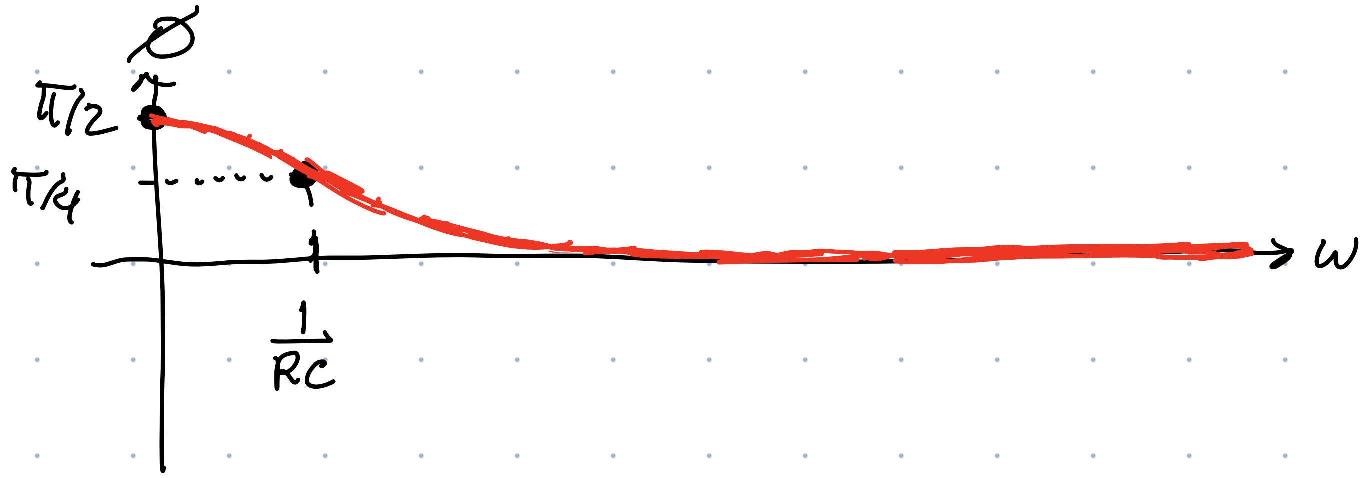
$$\therefore \tan\varnothing = \underbrace{\frac{\cancel{V_{in}}\, w\cancel{RC}}{\cancel{1+(wRC)^2}}}_{y} \; \underbrace{\frac{\cancel{1+(wRC)^2}}{\cancel{V_{in}}\,(wRC)^{\cancel{2}}}}_{\frac{1}{x}} = \frac{1}{wRC}$$

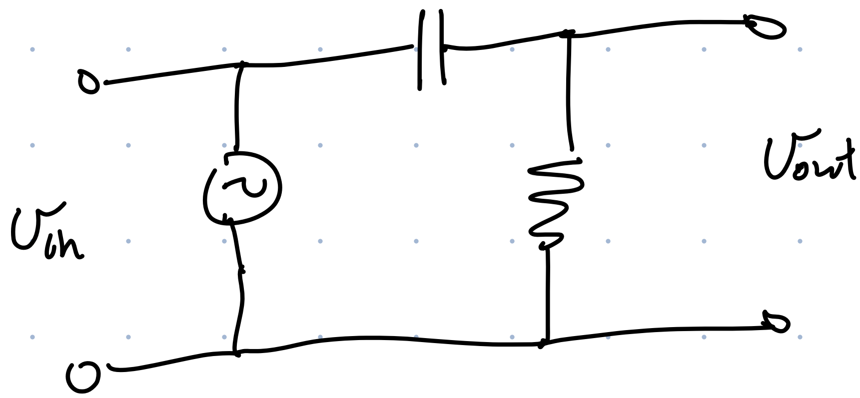
$$\boxed{\therefore \varnothing = \tan^{-1}\left(\frac{1}{wRC}\right)}$$

ω	$1/wRC$	$\varnothing$
0	∞	$\pi/2$
$\frac{1}{RC}$	1	$\pi/4$
∞	0	0





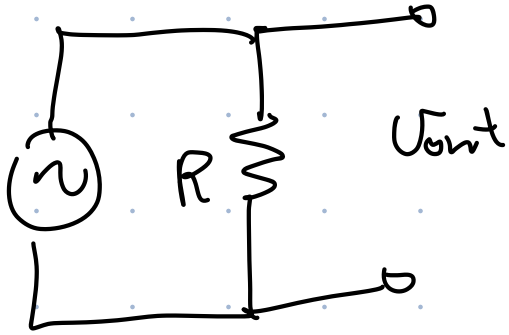




At high freq, no phase ($\phi = 0$) between input and output.

$$Z = \frac{1}{j\omega C} + R$$

At high freq $\left(\omega \gg \frac{1}{RC}\right)$, the impedance of the cap. vanishes $\left(Z_c = \frac{1}{j\omega C}\right)$ and the circuit looks like:

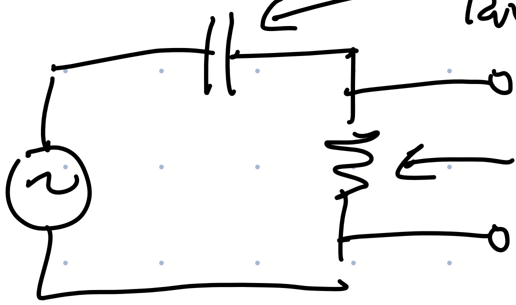


no phase,
no attenuation.

At low freq, $\left(\omega \ll \frac{1}{RC}\right)$ the impedance of the cap. becomes very large $\left(Z_C = \frac{1}{j\omega C}\right)$ s.t. $Z_C \gg R$

and $Z \approx \dfrac{1}{j\omega C}$

The cap. introduce the phase shift of 90°.



large impedance

small impedance at low ω