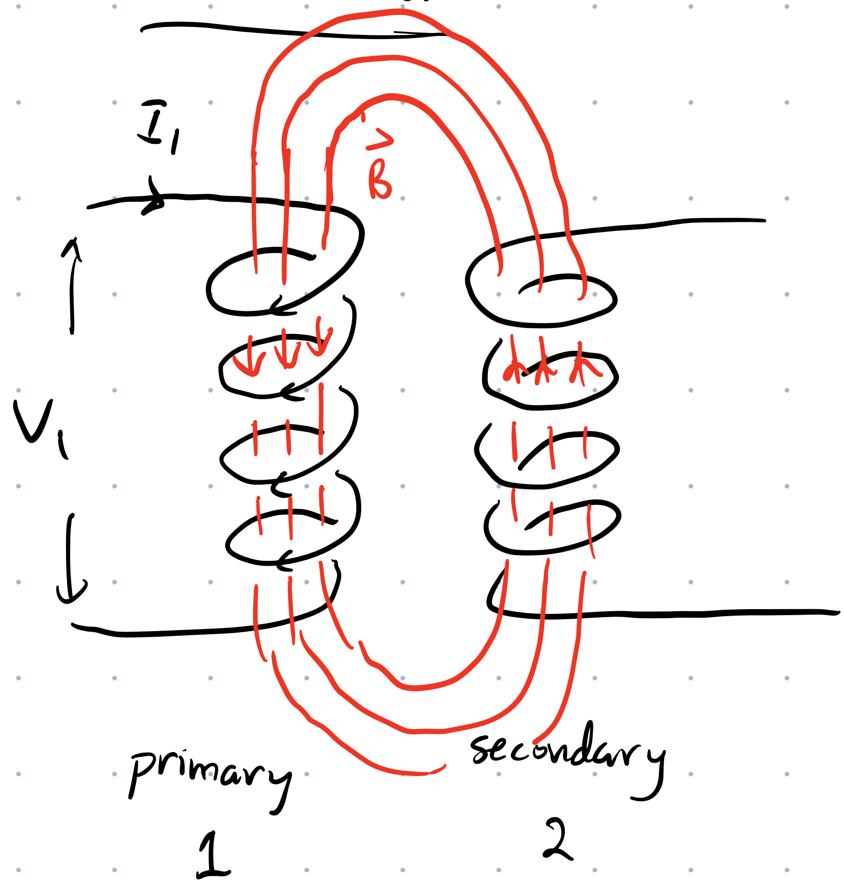
PHYS 331 Nov. 7, 2024

Transformers



Usually trans. windings are wrapped around a magnetic material (like iron).

$\Rightarrow$ All $\vec{B}$ fields that pass through primary also pass through secondary.

$\Phi_1 = \Phi_2$ magnetic flux

$\equiv \Phi$

Trans. works only for AC currents/voltages. b/c we need a changing magnetic flux to induce voltage/current in secondary.

Faraday's Law: $V_1 = -N_1 \dfrac{d\Phi}{dt}$

$V_2 = -N_2 \dfrac{d\Phi}{dt}$

N_1: no. of windings in primary

N_2: no. of windings in secondary.

$$\therefore \frac{V_2}{V_1} = \frac{\cancel{-}N_2 \cancel{d\Phi/dt}}{\cancel{-}N_1 \cancel{d\Phi/dt}} \Rightarrow \boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}}$$

$$\therefore V_2 = V_1 \frac{N_2}{N_1}$$

We can use a transformer to easily step up $\left(\frac{N_2}{N_1} > 1\right)$ or step down $\left(\frac{N_2}{N_1} < 1\right)$ the voltage @ secondary.

By conservation of energy

$$V_1 I_1 = V_2 I_2$$

$$(P_1 = P_2)$$

$$\frac{I_2}{I_1} = \frac{V_1}{V_2} = \frac{N_1}{N_2}$$

$$\Rightarrow \boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1} \qquad \frac{I_2}{I_1} = \frac{N_1}{N_2} \qquad \frac{P_2}{P_1} = 1}$$

Impedance

Primary $\quad Z_1 = \dfrac{V_1}{I_1}$

Secondary $\quad Z_2 = \dfrac{V_2}{I_2}$

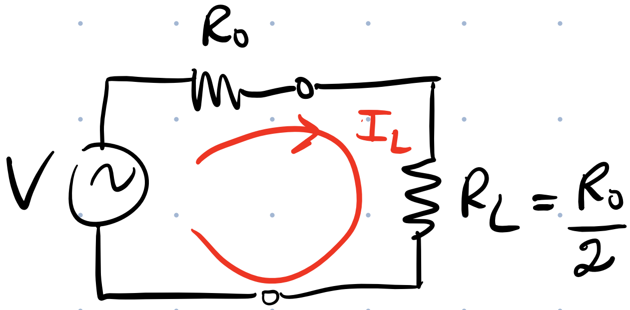
$$\frac{Z_2}{Z_1} = \frac{V_2}{I_2}\,\frac{I_1}{V_1}$$

$$= \left(\frac{V_2}{V_1}\right)\left(\frac{I_1}{I_2}\right) = \left(\frac{N_2}{N_1}\right)^2$$

$$\boxed{\frac{Z_2}{Z_1} = \left(\frac{N_2}{N_1}\right)^2}$$

Practical Applications

1. Impedance Matching.



$$I_L = \frac{V}{R_0 + R_L} = \frac{V}{3R_0/2}$$

$$I_L = \frac{2V}{3R_0}$$

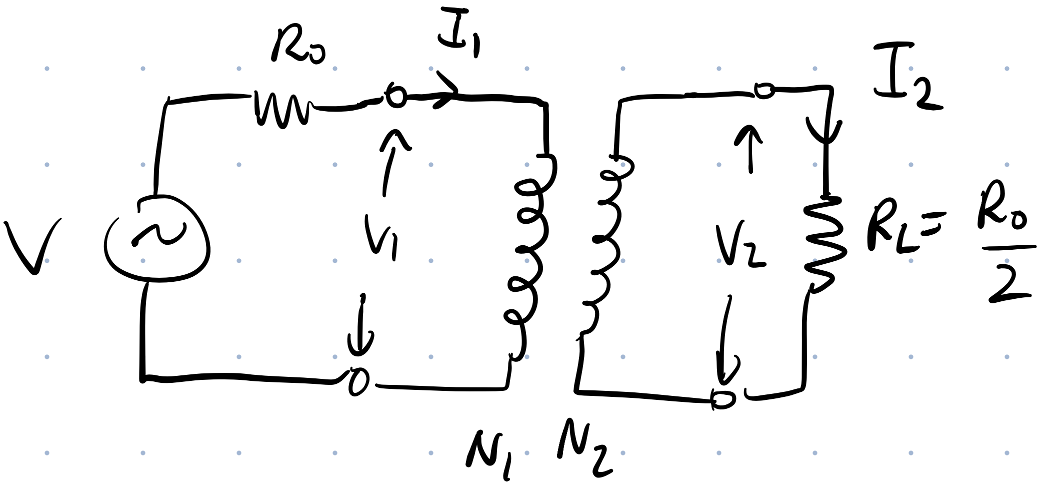
$$P_L = I_L^2 R_L = \frac{4}{9}\frac{V^2}{R_0^2}\left(\frac{R_0}{2}\right) = \frac{2}{9}\frac{V^2}{R_0}$$

Source power $P_s = V I_L = \dfrac{2V^2}{3R_0}$

Power Trans. Efficiency $\eta = \dfrac{P_L}{P_s} = \dfrac{\frac{2}{9}\cancel{\frac{V^2}{R_0}}}{\frac{2}{3}\cancel{\frac{V^2}{R_0}}} = \dfrac{\cancel{2}}{9}\dfrac{3}{\cancel{2}} = \dfrac{1}{3}$

In this example $\frac{1}{3}$ of system power is delivered to R_L & $\frac{2}{3}$ is lost to R_0 (output resistance of signal gen.).

Next, consider



choose $\left(\frac{N_2}{N_1}\right)^2 = \frac{1}{2}$

Know: $V_2 = I_2 R_L$

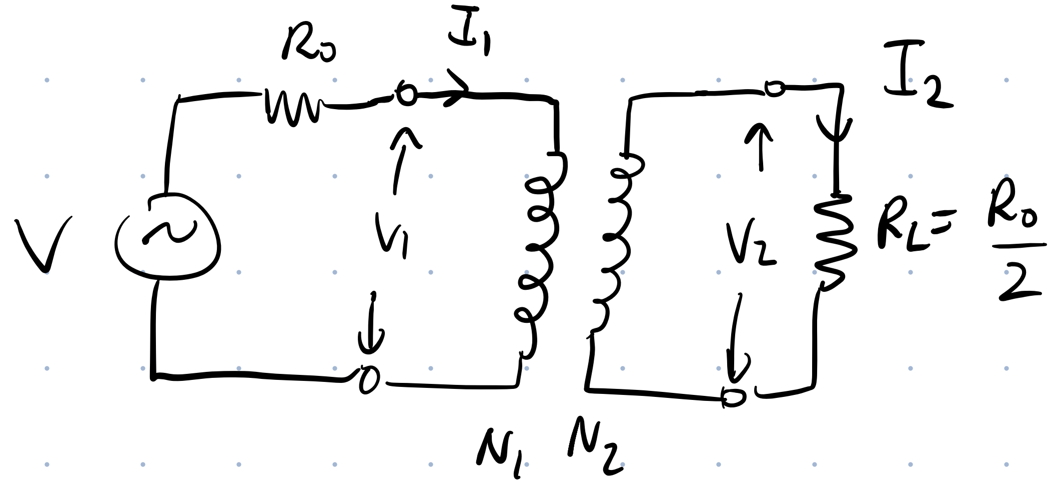
$$\therefore V_1\left(\frac{N_2}{N_1}\right) = I_1\left(\frac{N_1}{N_2}\right) R_L$$

$$\therefore V_1 = I_1 \left(\frac{N_1}{N_2}\right)^2 R_L$$

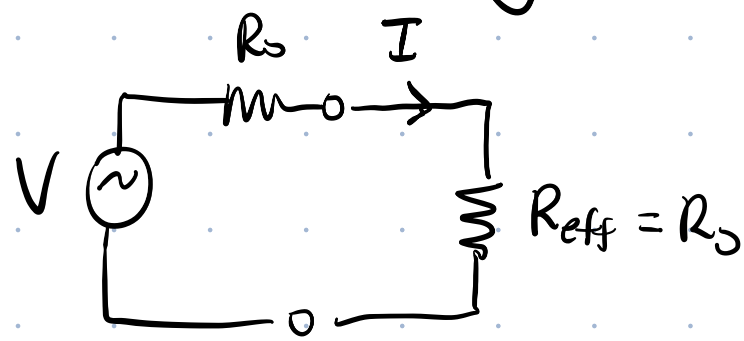
$$= I_1 \underbrace{\left[\left(\frac{N_1}{N_2}\right)^2 R_L\right]}_{R_{eff} = 2\left(\frac{R_0}{2}\right) = R_0}$$

We have used the transformer to transform the load resistance R_L from $\frac{R_0}{2}$ to $R_{eff} = R_0$ (impedance matching).

Equiv. circuit.



$\Downarrow$



current in equiv. circuit is $I = \frac{V}{2R_0}$

$$P_S = VI = \frac{V^2}{2R_0}$$

$$P_L = I_2^2 R_L = \left(I_1 \frac{N_1}{N_2}\right)^2 \frac{R_0}{2}$$

same!

Need to find I_1. Start w/.

$$V - I_1 R_0 = V_1$$

$$V = I_1 R_0 + I_1 \left(\frac{N_1}{N_2}\right)^2 R_L$$

Solve for $I_1 = \dfrac{V}{R_0 + \left(\frac{N_1}{N_2}\right)^2 R_L} = \dfrac{V}{R_0 + R_0} = \boxed{\dfrac{V}{2R_0}}$

$$\therefore P_L = \left(\frac{V}{2R_0}\frac{N_1}{N_2}\right)^2 \frac{R_0}{2} = \frac{V^2}{4R_0^{\cancel{2}}} \cancel{2} \frac{\cancel{R_0}}{\cancel{2}}$$

$$= \frac{V^2}{4R_0}$$

$\therefore$ The new power transfer efficiency is:

$$\eta = \frac{P_L}{P_S} = \frac{V^2/4R_0}{V^2/2R_0} = \frac{1}{2} \checkmark$$

Note that we could have analyzed the simpler equiv. circuit to find the same result.

$$P_s = VI = \frac{V^2}{2R_o}$$

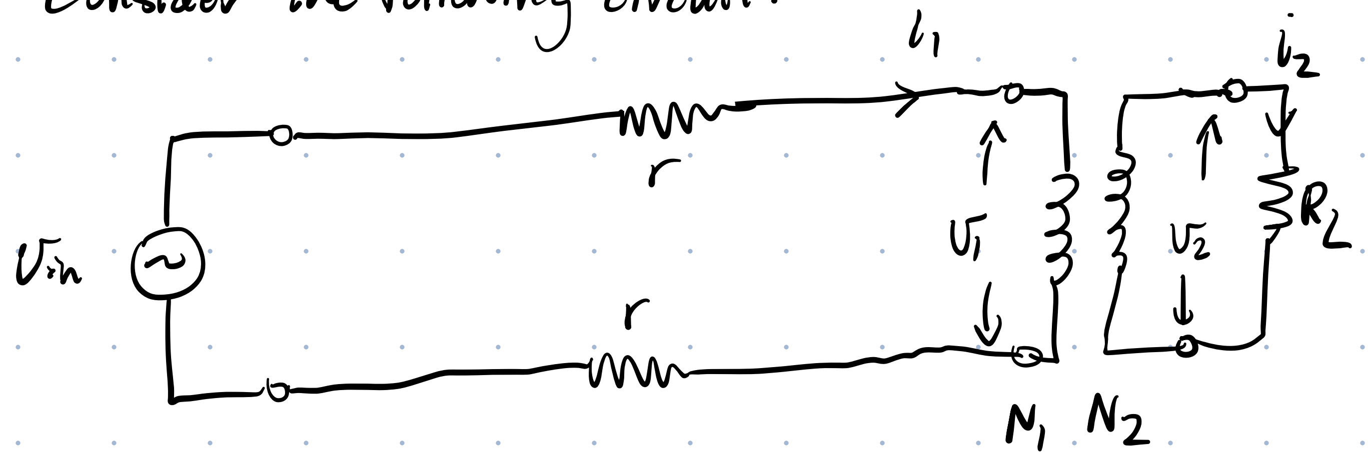
$$P_L = I^2 R_{eff} = \left(\frac{V}{2R_o}\right)^2 R_o = \frac{V^2}{4R_o}$$

$$\therefore \eta = \frac{P_s}{P_L} = \frac{1}{2}$$ same as before!

For a source w/ output resistance R_o, get a max. η of $1/2$ when $R_L = R_o$. Dissipated at least 50% of power in output resistance of signal generator.

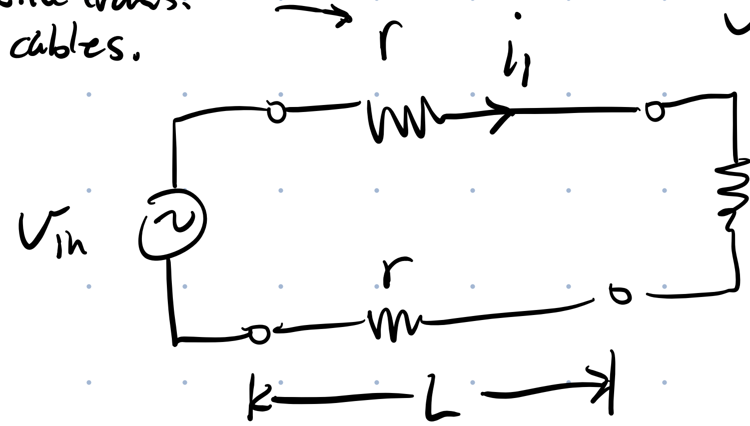
Question: Why do hydro lines deliver power at high voltage & low current?

Consider the following circuit.



⇓ equiv. circuit

small resistance assoc. with trans. cables. →



$R_{eff} = \left(\frac{N_1}{N_2}\right)^2 R_L$

Keep r & R_L fixed. $r = 0.5\Omega$

$R_L = 9\Omega$

Want to deliver power to R_L while minimizing the power dissipated by trans. line.

What about trans. line effects?

For power delivered to homes $f = 60\text{Hz}$.

$$\lambda = \frac{c}{f} = \frac{3\times10^8 \text{ m/s}}{60 \text{ 1/s}} = 0.5\times10^7 \text{m}$$
$$= 5000 \text{ km!}$$

Assume that $L \ll \lambda$

$$i_1 = \frac{v_{in}}{R_{eff}+2r} = \frac{v_{in}}{\left(\frac{N_1}{N_2}\right)^2 R_L + 2r}$$

Power Provided by sys. $\quad P_{sys} = v_{in} i_1 = \dfrac{v_{in}^2}{\left(\frac{N_1}{N_2}\right)^2 R_L + 2r}$

Power delivered to R_L $\quad P_L = i_2^2 R_L = i_1^2 \left(\dfrac{N_1}{N_2}\right)^2 R_L$

Power dissipated by trans. line $\quad P_r = 2 i_1^2 r$

Plan is to vary v_{in} & $\frac{N_1}{N_2}$ while keeping

$P_{sys} = v_{in} i_1 = \text{const.}$

Eg. $\quad v_{in} = 1000V \quad \frac{N_1}{N_2} = 1$

In this case, $i_1 = \dfrac{1000V}{(1)^2 9\Omega + 2(0.5\Omega)} = 100A.$

Low volt. (relatively) & high current case.

$P_{sys} = v_{in} i_1 = 100 \text{ kW}$

$P_L = (100A)^2 (1)^2 9\Omega = 90 \text{kW}$

$P_r = 2(100A)^2 (0.5\Omega) = 10 \text{kW}$

In this case, $\frac{P_r}{P_{sys}} = \frac{10}{100} \Rightarrow$ 10% power loss to dissipation by trans. line.

Let's now consider the high voltage, low current case. Keep $P_{sys} = 100\,kW$, but set $v_{in} = 100\,kV$

$$\therefore\ i_1 = \frac{P_{sys}}{v_{in}} = \frac{100\,kW}{100\,kV} = 1\,A$$

Figure out $\frac{N_1}{N_2}$: $\quad i_1 = \dfrac{v_{in}}{\left(\frac{N_1}{N_2}\right)^2 R_L + 2r}$

$$\therefore \left(\frac{N_1}{N_2}\right) = \sqrt{\frac{\frac{v_{in}}{i_1} - 2r}{R_L}} = 105.4087\ldots$$

$$P_L = i_1^2 \left(\frac{N_1}{N_2}\right)^2 R_L = 99.999 \text{ kW}$$

$$P_r = 2 i_1^2 r = 1 \text{W} = 0.001 \text{ kW}$$

$$\frac{P_r}{P_{sys}} = \frac{1}{10^5} \Rightarrow 0.001\%$$

power dissipated by cables.