PHYS 231 Oct. 24, 2024

Last Time:

Convolution Theorem:

If $\hat{y}(\omega) = \hat{x}_1(\omega)\,\hat{x}_2(\omega)$,

then $y(t) = F^{-1}\left[\hat{y}(\omega)\right] = (x_1 * x_2)(t)$

where:

$$(x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

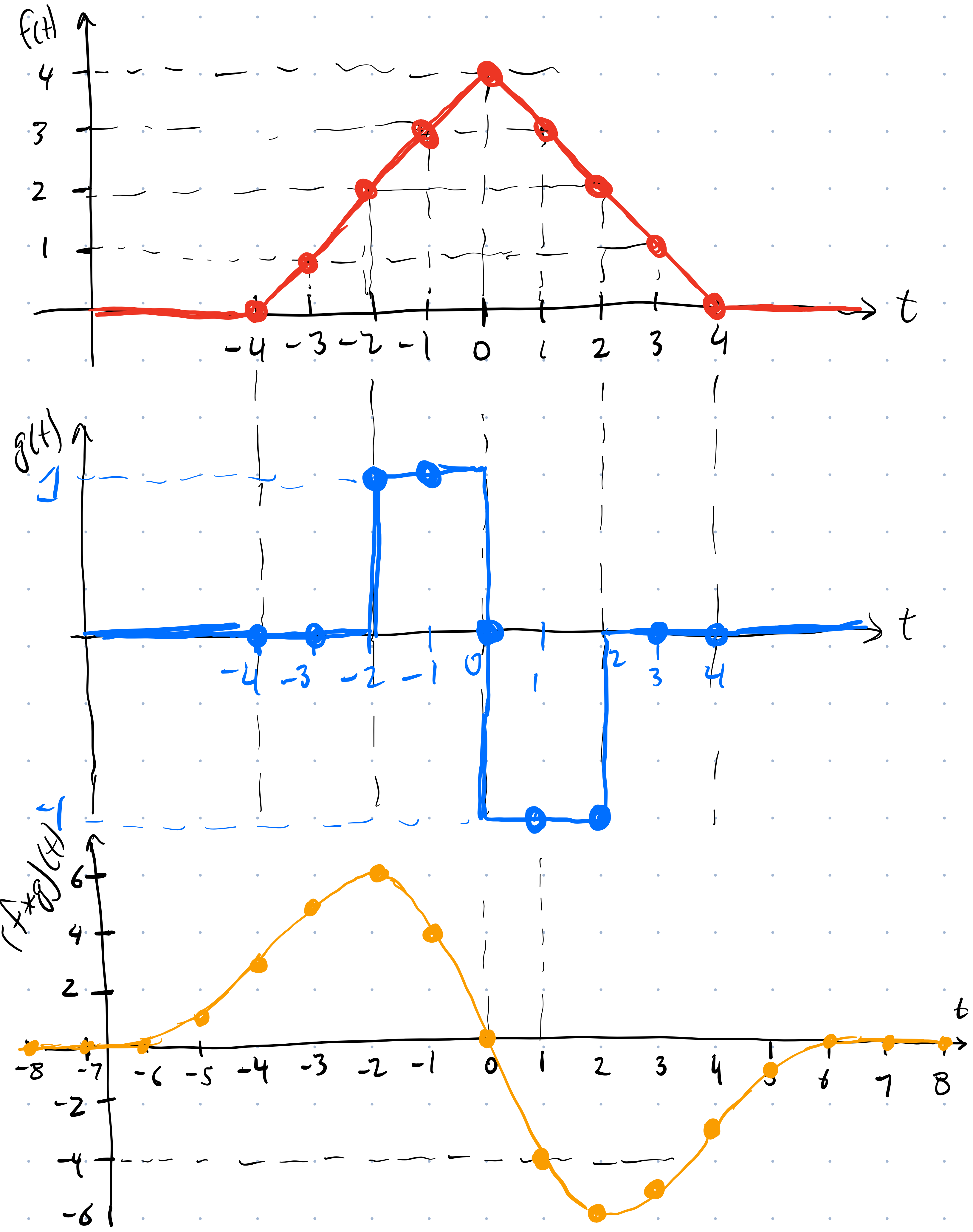
Today: Manually evaluate a convolution of two fcns to try to gain insight into what the convolution does.

- Example Python code that reproduces our calculation

- Convince you that, without knowing it, you've been evaluting convolutions for a long time.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

Let's pick a $f(t)$ & $g(t)$ to work with.


f(t)
g(t)
(f*g)(t)
t

Since our fcns are non-zero only on $-T < t < T$, where $T = 4$ we can write:

$$(f * g)(t) = \int_{-T}^{T} f(\tau)\, g(t-\tau)\, d\tau$$

Approx integral as a sum

$$(f * g)(t) = \sum_{\tau=-T}^{T} f(\tau)\, g(t-\tau)\, \Delta\tau$$

set $\tau = n\Delta\tau$ $\qquad n = -N \dots N$

where, in our ex., $N = 4$.

$$\Delta\tau = \frac{T}{N}$$

$$(f * g)(t) \simeq \sum_{n=-N}^{N} f(n\Delta\tau)\, g(t - n\Delta\tau)\, \Delta\tau$$

Make a table of values of our discretized fcns.

domain: $-4 < \tau < 4$

$\rightsquigarrow$

$-4 < n < 4$

$T = 4$

$N = 4$

$\Delta\tau = \frac{T}{N} = 1$

n	$f(n\Delta\tau)$	$g(n\Delta\tau)$
-4	0	0
-3	1	0
-2	2	1
-1	3	1
0	4	0
1	3	-1
2	2	-1
3	1	0
4	0	0

Write out the terms of the sum for our example case:

$\Delta\tau = 1$

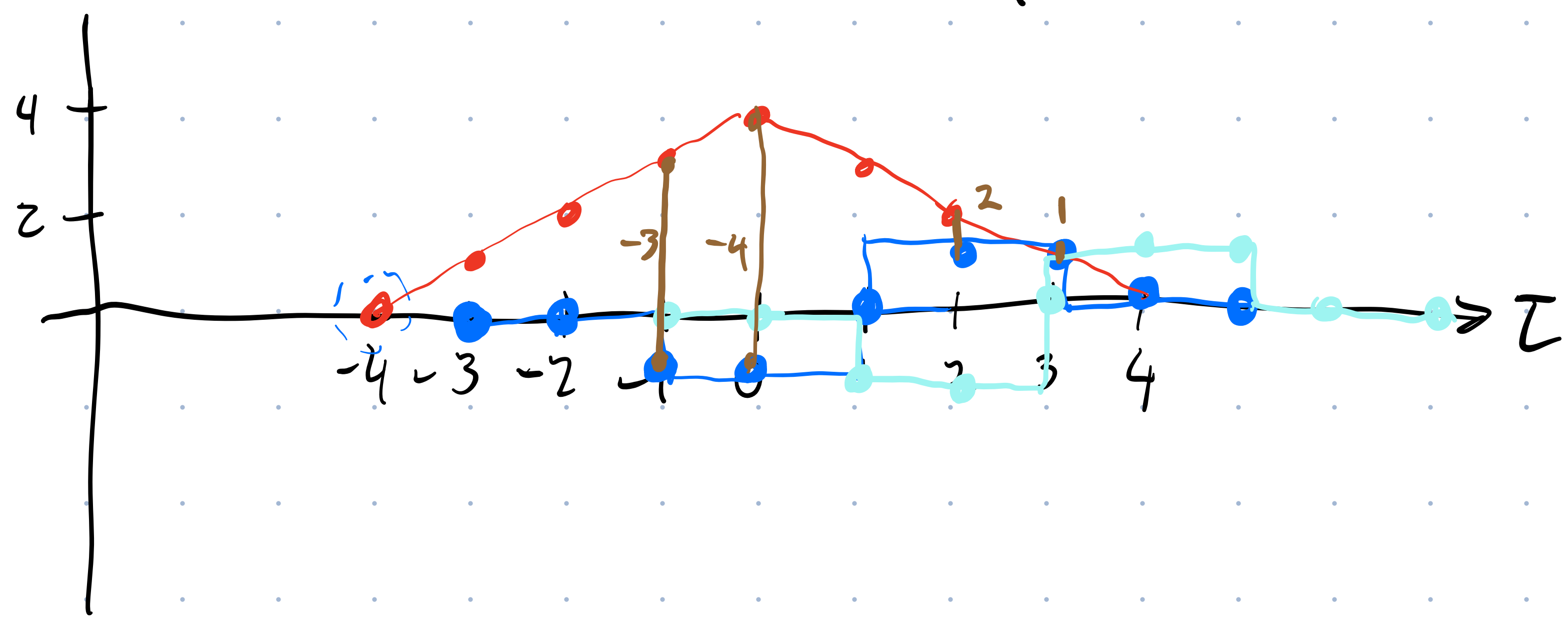
$$(f * g)(t) = \sum_{n=-4}^{4} f(n)\, g(t-n)$$

$$= f(-4)g(t+4) + f(-3)g(t+3)$$
$$+ f(-2)g(t+2) + f(-1)g(t+1)$$
$$+ f(0)g(t) + f(1)g(t-1)$$
$$+ f(2)g(t-2) + f(3)g(t-3) + f(4)g(t-4)$$

Try the $t=1$ case:

$$(f * g)(1) = \overset{0}{f(-4)}\,\overset{?}{g(5)} + \overset{1}{f(-3)}\,\overset{0}{g(4)}$$
$$+ \overset{2}{f(-2)}\,\overset{0}{g(3)} + \overset{3}{f(-1)}\,\overset{-1}{g(2)}$$
$$+ \overset{4}{f(0)}\,\overset{-1}{g(1)} + \overset{3}{f(1)}\,\overset{0}{g(0)}$$
$$+ \overset{2}{f(2)}\,\overset{1}{g(-1)} + \overset{1}{f(3)}\,\overset{1}{g(-2)}$$
$$+ \overset{0}{f(4)}\,\overset{0}{g(-3)}$$

$$= -3 - 4 + 2 + 1 = -4$$



Notice that:

- g is shifted by an amount t.
- g is flipped $\Rightarrow g(t-\tau)$

Try the $t=3$ case.

$$(f*g)(3) = \overset{0}{f(-4)}\overset{?}{g(7)} + \overset{1}{f(-3)}\overset{?}{g(6)} + \overset{2}{f(-2)}\overset{?}{g(5)}$$
$$+ \overset{3}{f(-1)}\overset{0}{g(4)} + \overset{4}{f(0)}\overset{0}{g(3)} + \overset{3}{f(1)}\overset{-1}{g(2)}$$
$$+ \overset{2}{f(2)}\overset{-1}{g(1)} + \overset{1}{f(3)}\overset{0}{g(0)} + \overset{0}{f(4)}\overset{1}{g(-1)}$$

$$= g(6) + 2g(5) - 3 - 2 = -5$$

Assume that outside original domain $-4 < t < 4$, that $f(t)$ & $g(t)$ are zero.

Repeat this process $\forall$ t to find $(f*g)(t)$.

n	$(f*g)(n)$	t_{max}
-8	0	
-7	0	
-6	0	
-5	1	
-4	3	-8
-3	5	-7
-2	6	-6
-1	4	-5
0	0	N/A
1	-4	5
2	-6	6
3	-5	7
4	-3	8
5	-1	
6	0	
7	0	
8	0	

Things to note:

- $(f*g)(t)$ doubles the original domain size, extending it equally at each end.

- Like smoothing fcn.

- At each value of t, evaluate the inner product between $f(\tau)$ & $\underbrace{g(t-\tau)}_{\text{g is flipped \& shifted.}}$

To demonstrate that you've been evaluating convolutions for a long time, consider, for some reason, constructing polynomials using our sampled data pts as coefficients

$$f \rightarrow (0x^{-4} + 1x^{-3} + 2x^{-2} + 3x^{-1} + 4 + 3x^{1} + 2x^{2} + 1x^{3} + 0x^{4}) \cdot (0x^{-4} + 0x^{-3} + 1x^{-2} + 1x^{-1} + 0 - 1x - 1x^{2} + 0x^{3} + 0x^{4}) \leftarrow g$$

$$\left(x^{-3} + 2x^{-2} + 3x^{-1} + 4 + 3x + 2x^2 + x^3\right)$$
$$\cdot \left(x^{-2} + x^{-1} - x - x^2\right)$$

$$\boxed{x^{-5}} + \boxed{x^{-4}} \boxed{- x^{-2}} \boxed{- x^{-1}}$$
$$+ \boxed{2x^{-4}} + \boxed{2x^{-3}} \boxed{-2x^{-1}} \boxed{-2}$$
$$+ \boxed{3x^{-3}} \boxed{+ 3x^{-2}} \boxed{-3} - 3x$$
$$+ \boxed{4x^{-2}} \boxed{+ 4x^{-1}} - 4x - 4x^2$$
$$+ \boxed{3x^{-1}} \boxed{+ 3} - 3x^2 - 3x^3$$
$$\boxed{+ 2} + 2x - 2x^3 - 2x^4$$
$$+ x + x^2 - x^4 - x^5$$

$$= \boxed{0} \cdot x^{-8} + \boxed{0} \cdot x^{-7} + \boxed{0} \cdot x^{-6}$$
$$\boxed{1} \cdot x^{-5} + \boxed{3} \cdot x^{-4} + \boxed{5} x^{-3} + \boxed{6} x^{-2} + \boxed{4} x^{-1} + \boxed{0}$$
$$- \boxed{4} x - \boxed{6} x^2 - \boxed{5} x^3 - \boxed{3} x^4 - \boxed{1} x^5$$
$$+ \boxed{0} x^6 + \boxed{0} x^7 + \boxed{0} x^8$$

coefficients are $(f * g)$!