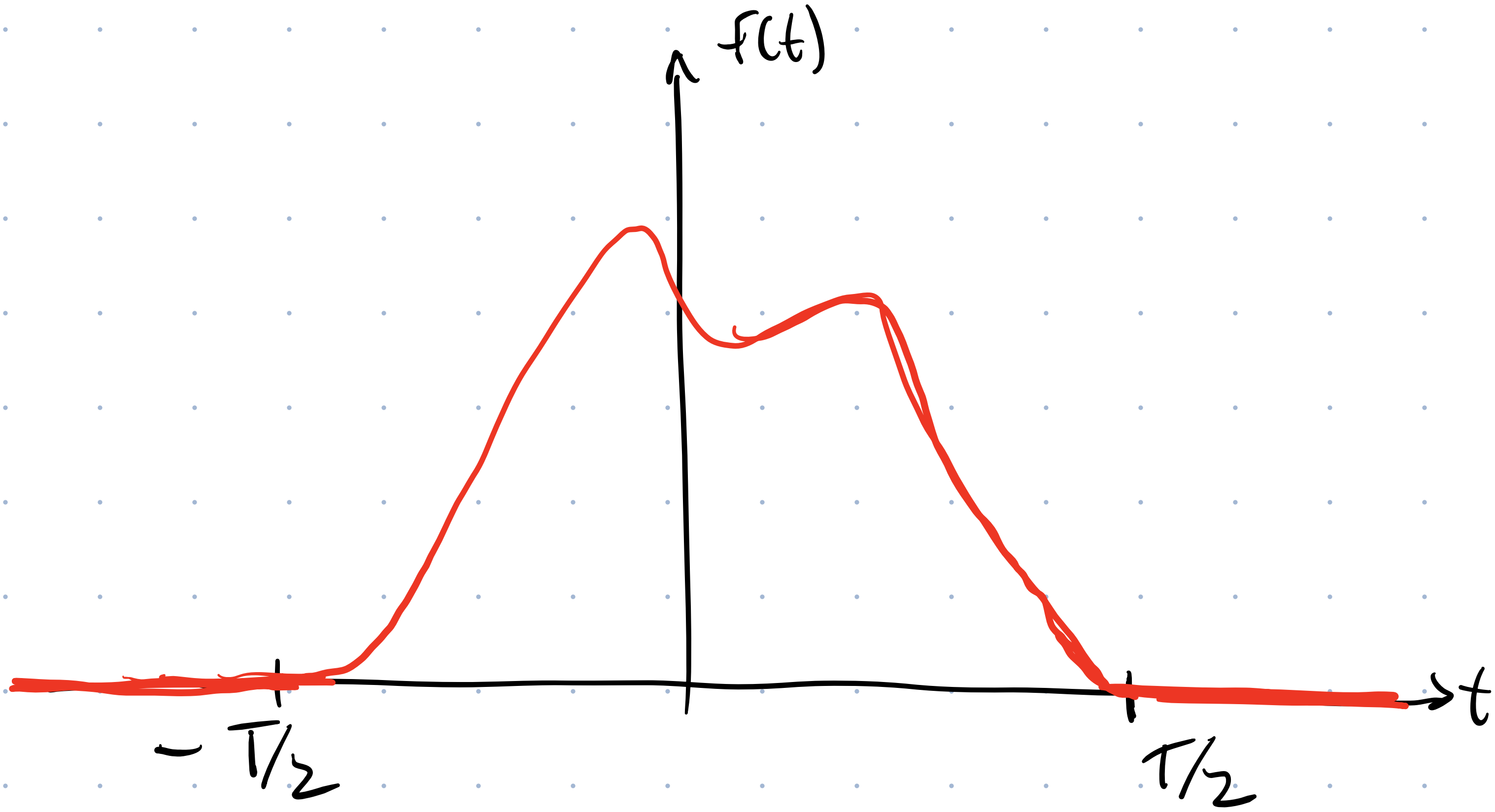
PHYS 331

Oct. 17, 2024

Last Time:

Generalize the Fourier series so that we can examine freq. content of a signal $f(t)$ that is a pulse, not periodic.

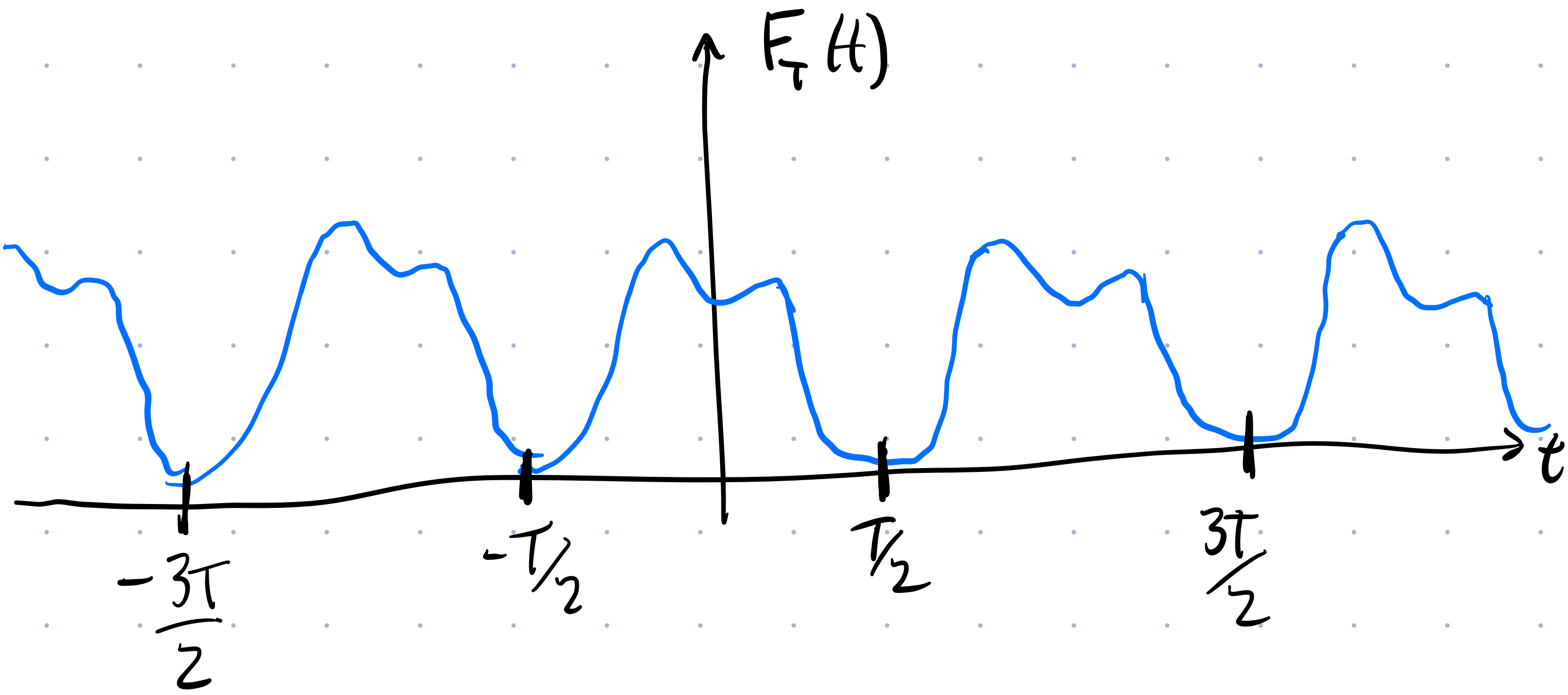
Eg. $f(t) = 0 \quad \forall \; |t| \geq \frac{T}{2}$



Construct a periodic fcn using this $f(t)$ pulse.

$$F_T(t) = f(t) \quad \text{on} \quad -\frac{T}{2} < t < \frac{T}{2}$$

our constructed periodic fcn



Then, for $\boxed{-T/2 < t < T/2}$

$$f(t) = F_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t}$$

$\left(\omega = \frac{2\pi}{T}\right)$

periodic $F_T(t)$ can be written as a Fourier series

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} F_T(t)e^{-jn\frac{2\pi}{T}t}$$

Found:

$$c_n = \underbrace{\frac{\Delta\omega}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega_n t}\, dt}_{\hat{f}(\omega_n)\ (!)}$$

where $\Delta\omega = \frac{2\pi}{T}$ & $\omega_n = \frac{2\pi n}{T}$

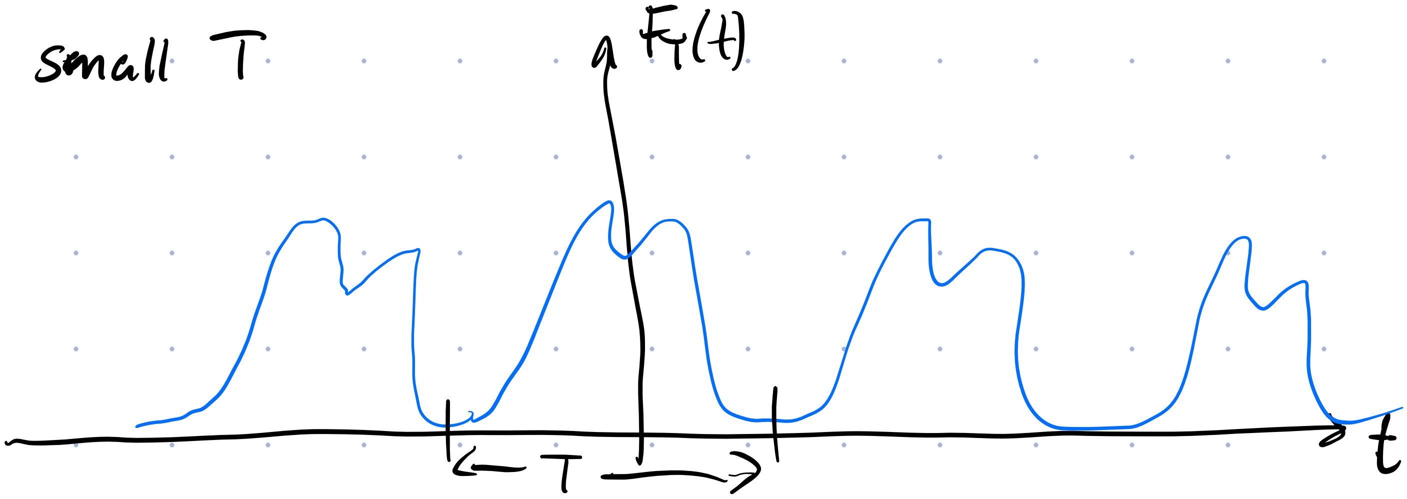
$$\therefore\ c_n = \frac{1}{2\pi}\hat{f}(\omega_n)\,\Delta\omega$$

Return to (*):

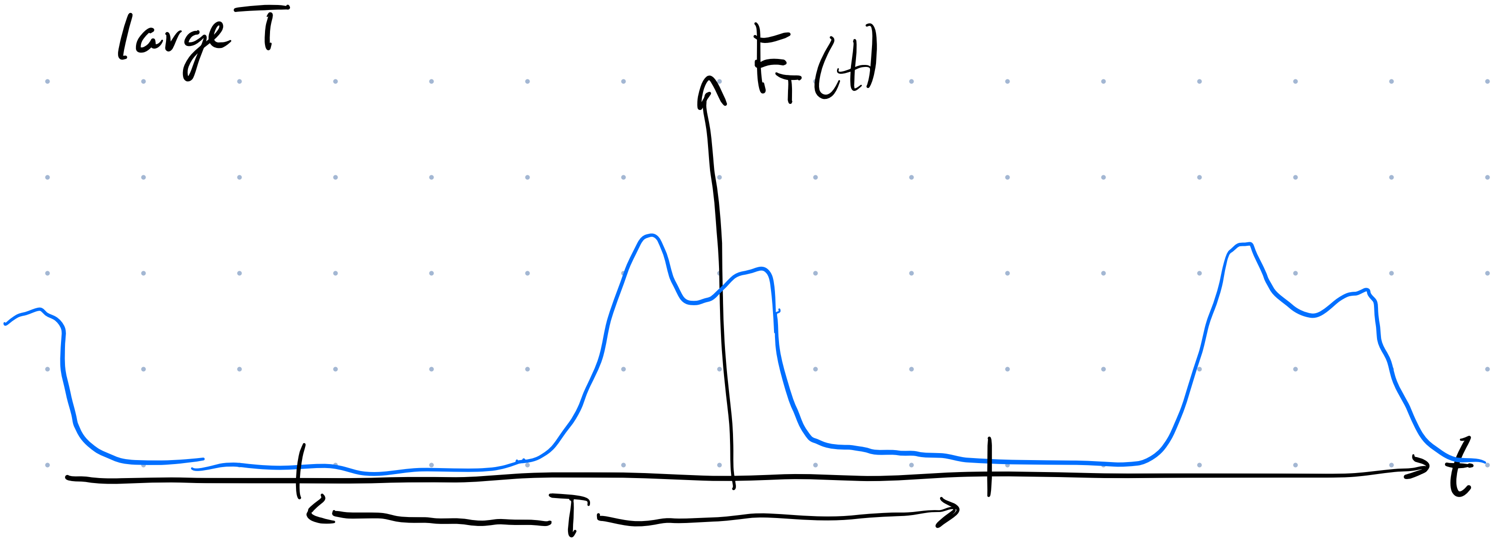
On $-T/2 < t < T/2$ we have

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\hat{f}(\omega_n)\, e^{j\omega_n t}\,\Delta\omega \quad (\#)$$

small T



large T



In the limit $T \to \infty$ # becomes

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{j\omega t}\, d\omega}$$

Inverse Fourier transform of $\hat{f}(\omega)$

→ It takes a fcn of freq ($\hat{f}(\omega)$) and outputs a fcn of time ($f(t)$).

$\Delta\omega = \frac{2\pi}{T}$ $\Delta\omega \to 0$ as $T \to \infty$

$\Delta\omega$ was spacing between adjacent freq. components. If $\Delta\omega \to 0$, then $\omega_n = \frac{2\pi n}{T}$ becomes a continuous variable $\omega_n \to \omega$.

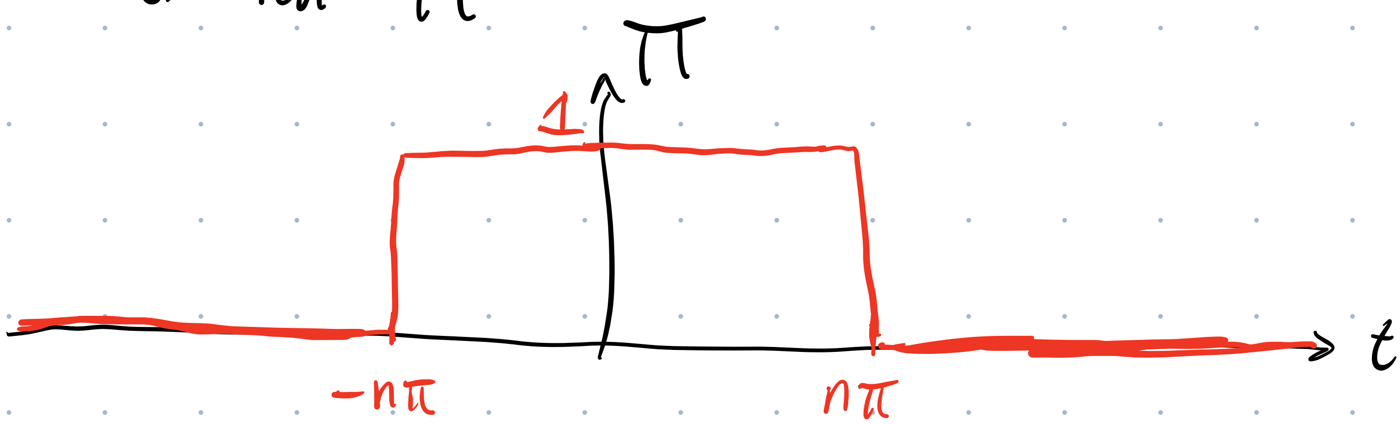
$$\boxed{\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt}$$

Fourier transform of $f(t)$.

It takes a fcn of time $(f(t))$ and converts it to a fcn of freq $(\hat{f}(\omega))$.

Fourier Transform Example

Box Fcn Π



$$\Pi(t) = \begin{cases} 1 & -n\pi < t < n\pi \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{\Pi}(\omega) = \int_{-\infty}^{\infty} \Pi(t) e^{-j\omega t} \, dt$$

$$= \int_{-n\pi}^{n\pi} 1 \cdot e^{-j\omega t} \, dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-n\pi}^{n\pi}$$

$$= -\frac{1}{j\omega}\left[e^{-jn\omega\pi} - e^{+jn\omega\pi}\right]$$

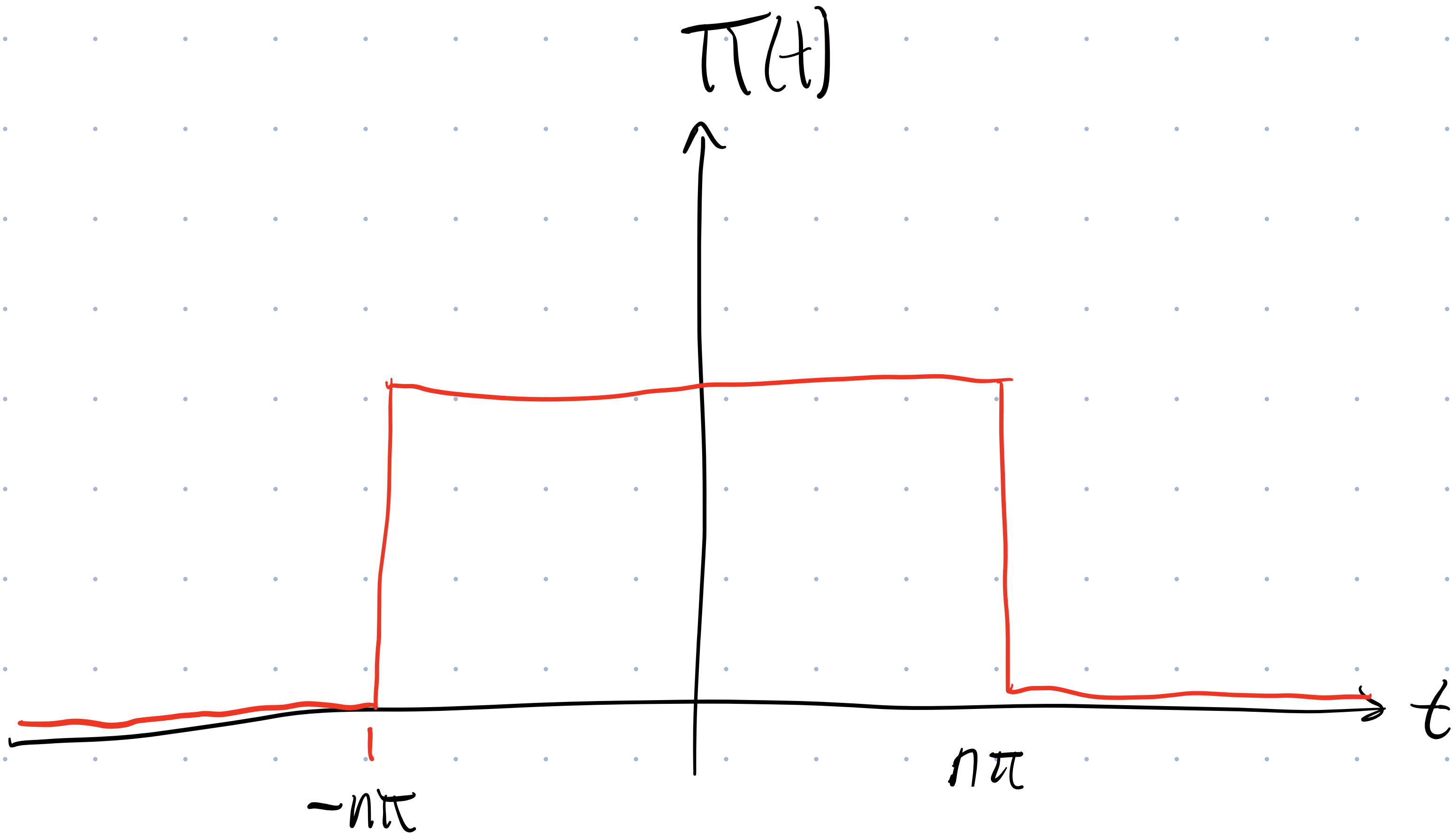
$$= + \frac{1}{\cancel{j}\omega}\left(\cancel{+}2\cancel{j} \sin n\omega\pi\right)$$

$$\hat{\Pi}(\omega) = \frac{2}{\omega} \sin n\omega\pi \; \frac{n\pi}{n\pi}$$

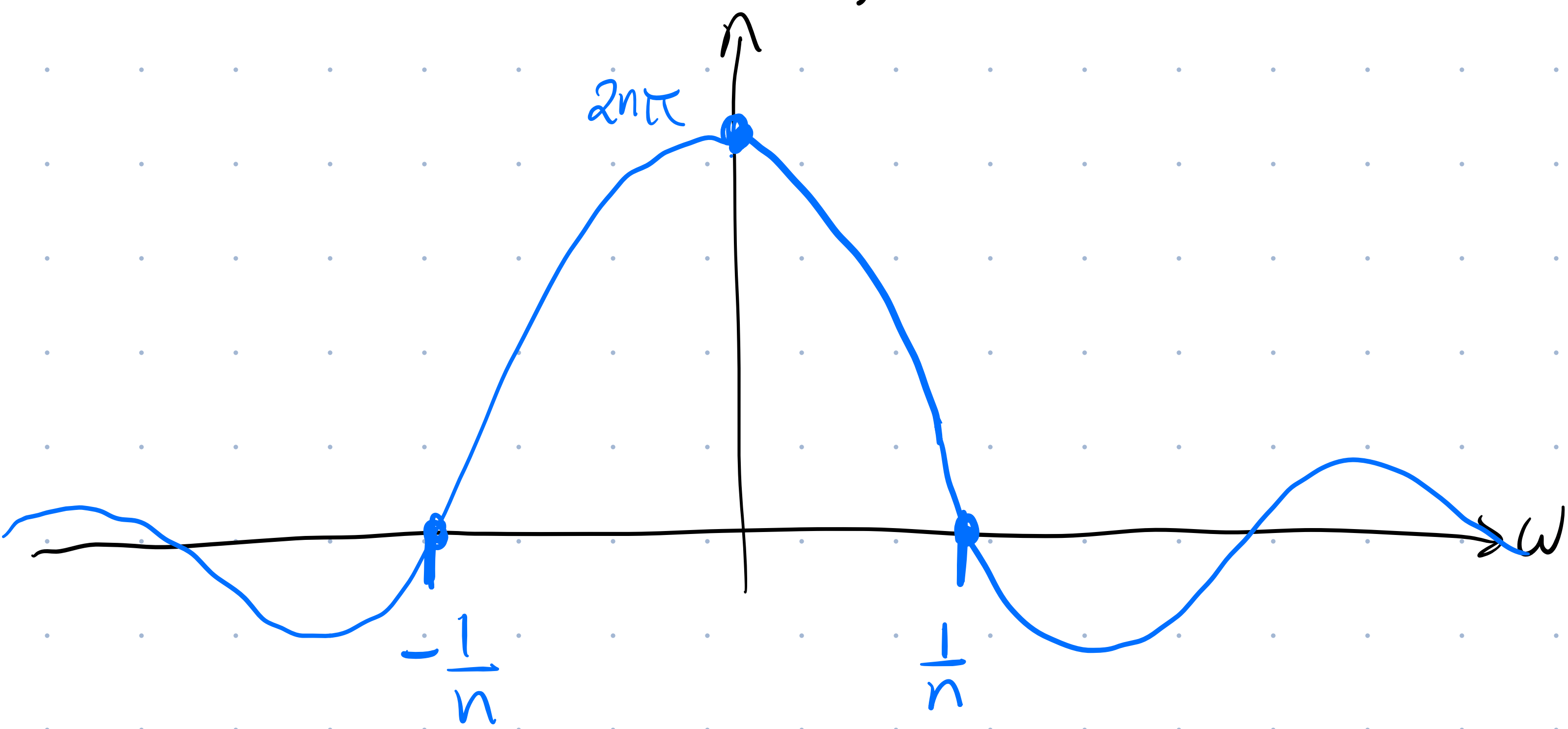
$$= 2n\pi \underbrace{\frac{\sin n\omega\pi}{n\omega\pi}}_{\mathrm{sinc}(n\omega)}$$

$$\hat{\Pi}(\omega) = 2n\pi \,\mathrm{sinc}(n\omega)$$

Plot $\Pi(t)$ & $\hat{\Pi}(\omega)$



$$\hat{\Pi}(\omega) = 2n\pi \operatorname{sinc}(n\omega)$$



Find the first zeros of sinc fcn.

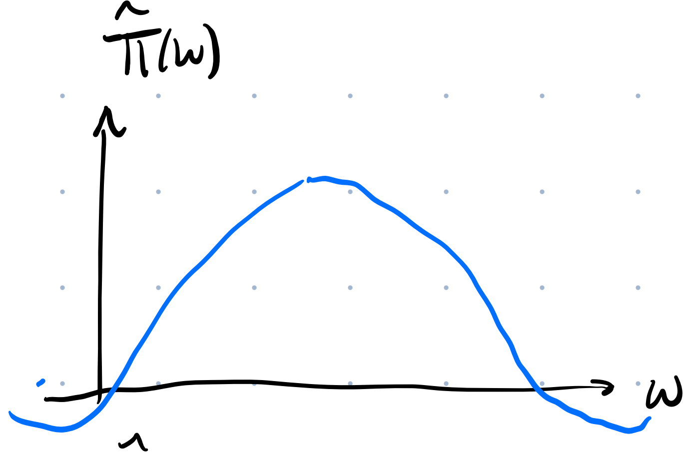
$$n\cancel{\pi}\omega = \pm \cancelto{1}{\pi}$$

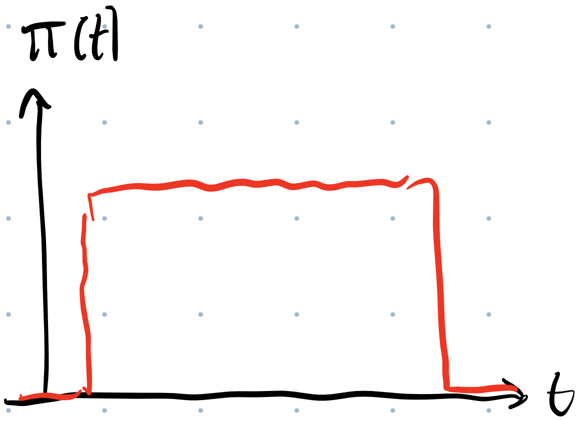
$$\therefore \quad n\omega = \pm 1$$

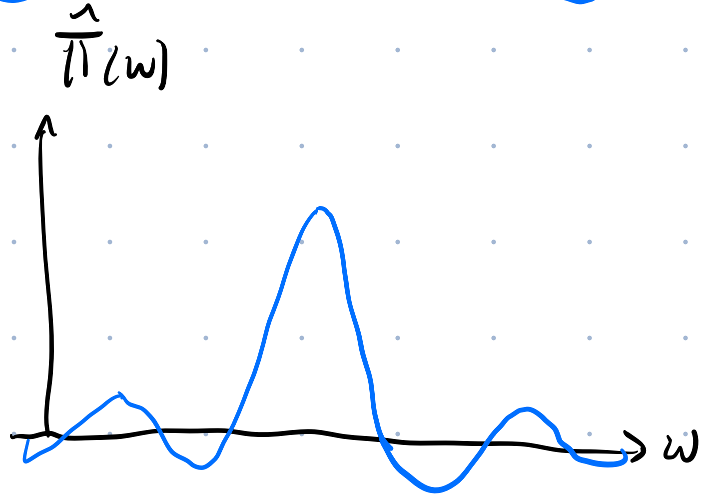
$$\omega = \pm \frac{1}{n}$$

Large n $\rightarrow$ wide fcn in time
narrow fcn in ω

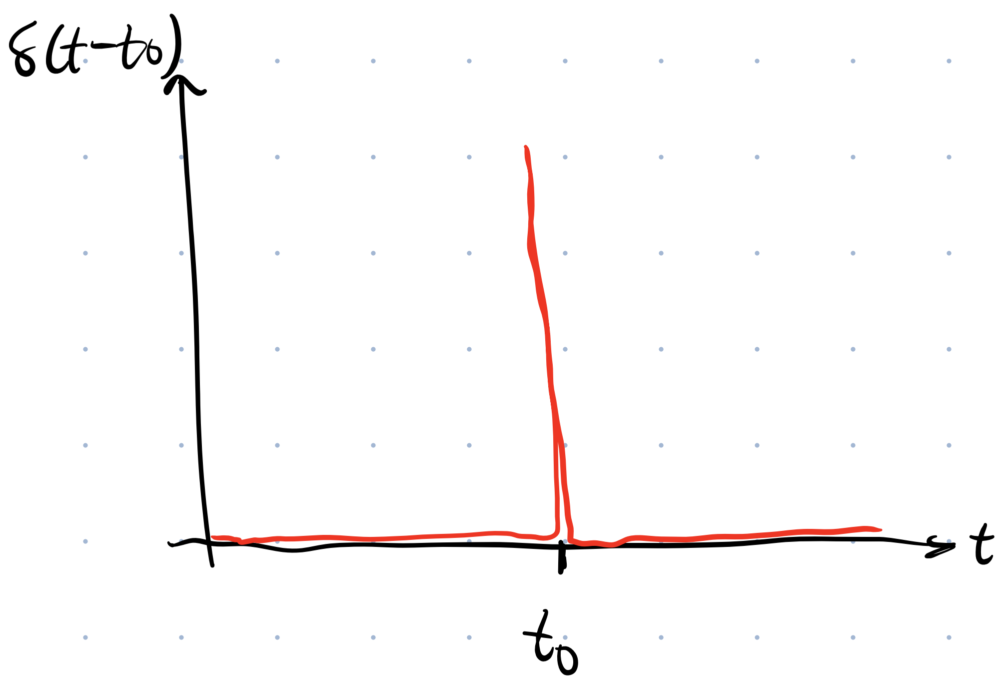
Small n $\rightarrow$ narrow in ω
wide in time.







Consider the extreme case of a δ-fcn → $\delta(t-t_0)$

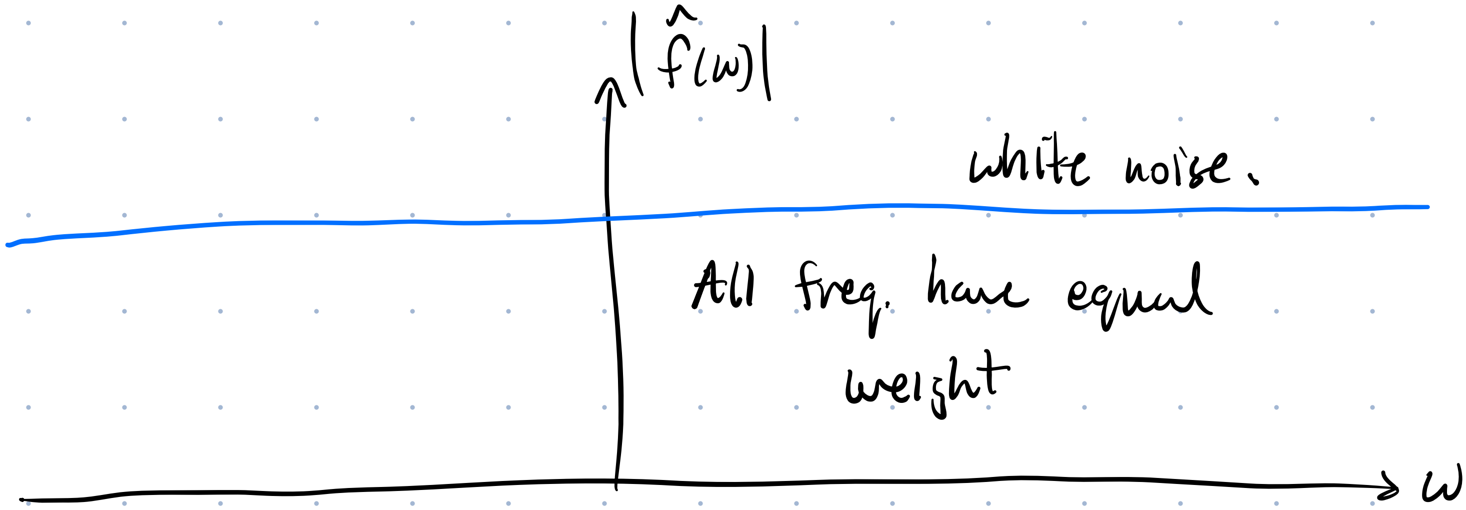


$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\omega t}\, dt$$

selects $e^{-j\omega t}\Big|_{t=t_0}$

$$\boxed{\therefore \ \hat{f}(\omega) = e^{-j\omega t_0}}$$

Mag. of $|\hat{f}(\omega)| = \sqrt{e^{-j\omega t_0} \, e^{+j\omega t_0}} = 1.$



Consider the phase of $\hat{f}(\omega) = e^{-j\omega t_0}$

$$\underset{\text{mag. } |\hat{f}(\omega)|}{1} \ \underbrace{e^{-j\omega t_0}}_{e^{j\phi}} \qquad \therefore \underbrace{\phi = -\omega t_0}_{\text{freq. dep. phase.}}$$

Consider the inverse Fourier transform of
$\hat{f}(w) = e^{-jwt_0}$ $\Rightarrow$ must return $\delta(t-t_0)$

$$f(t) = \delta(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\hat{f}(w)}_{e^{-jwt_0}}\, e^{jwt}\, dw$$

$$\boxed{\delta(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jw(t-t_0)}\, dw}$$

One way of mathematically defining the delta fcn.

Properties of Fourier Transforms (will prove these in Assign. #3)

1. If $g(t) = f(t+b)$

then $\hat{g}(w) = e^{jwb}\,\hat{f}(w)$ — Translation property

where $\hat{f}(w)$ is F.T. of $f(t)$.

2. If $g(t) = f(at)$ Scaling Property

then $\hat{g}(\omega) = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)$

3. $$\underbrace{F\left[\dot{f}(t)\right]}_{\text{F.T. of } \frac{df(t)}{dt}} = \int_{-\infty}^{\infty} \dot{f}(t) e^{-j\omega t} dt$$

$$= j\omega \hat{f}(\omega)$$

One additional property of Fourier transforms:
<u>Convolution Property</u>.

Suppose $\hat{y}(\omega) = \hat{x}_1(\omega)\, \hat{x}_2(\omega)$

i.e. $\hat{y}$ can be expressed as a product of two fcns of ω in which:

$$x_1(t) = F^{-1}\left[\hat{x}_1(\omega)\right]$$
$$x_2(t) = F^{-1}\left[\hat{x}_2(\omega)\right]$$

What is the inverse F.T. of $\hat{y}(\omega)$?

$$y(t) = F^{-1}\left[\hat{y}(\omega)\right]$$

$$= F^{-1}\left[\hat{x}_1(\omega)\,\hat{x}_2(\omega)\right] = ?$$

<u>Ans.</u>

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

These two integrals are equiv. & related by a change of variables

The convolution has a special notation:

$$x_1 * x_2 = (x_1 * x_2)(t)$$

<u>not</u> multiplication. It is the convolution of $x_1(t)$ & $x_2(t)$.

Convolution Theorem:

if $\hat{y}(\omega) = \hat{x}_1(\omega)\,\hat{x}_2(\omega)$

then $y(t) = (x_1 * x_2)(t)$

<u>Proof:</u> Strategy is to start w/ $y(t) = x_1 * x_2$, then take F.T. & show that $\hat{y}(\omega) = \hat{x}_1(\omega)\,\hat{x}_2(\omega)$.

$$y(t) = \underbrace{\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau}$$

$$\therefore\ \hat{y}(\omega) = \int_{-\infty}^{\infty} \underbrace{y(t)}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

Switch order of integration. Valid provided anything taken out of dt integral does not have a dependence on time.

$$\therefore \hat{y}(\omega) = \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt\, d\tau$$

make sub. $u = t - \tau$ in dt integral

$$du = dt \qquad t = u + \tau$$

$$\hat{y}(\omega) = \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(u)\, \underbrace{e^{-j\omega(u+\tau)}}_{e^{-j\omega u}\, e^{-j\omega \tau}}\, du\, d\tau$$

$$\hat{y}(\omega) = \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(u)\, e^{-j\omega u}\, e^{-j\omega \tau}\, du\, d\tau$$

$$\therefore \hat{y}(\omega) = \underbrace{\int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega \tau}\, d\tau}_{\hat{x}_1(\omega)} \underbrace{\int_{-\infty}^{\infty} x_2(u)\, e^{-j\omega u}\, du}_{\hat{x}_2(\omega)}$$

$$\therefore \quad \boxed{\hat{y}(\omega) = \hat{x}_1(\omega)\,\hat{x}_2(\omega)} \qquad \text{QED}$$

The inverse F.T. of the product $\hat{x}_1(\omega)\,\hat{x}_2(\omega)$ is the convolutional of $x_1(t)$ & $x_2(t)$.

An exercise for the student:

Suppose $y(t) = x_1(t)\,x_2(t)$

What is $\hat{y}(\omega) = F\left[y(t)\right]$?

Ans: $\hat{y}(\omega) = \frac{1}{2\pi}\left(\hat{x}_1 * \hat{x}_2\right)(\omega)$