PHYS 331 Oct. 10, 2024

Recall Fourier Series

- can represent any periodic fcn as an infinite series of sines & cosines.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t) \right]$$

↗ some arbitrary periodic fcn.

$\omega = \frac{2\pi}{T}$ T is period of f(t)

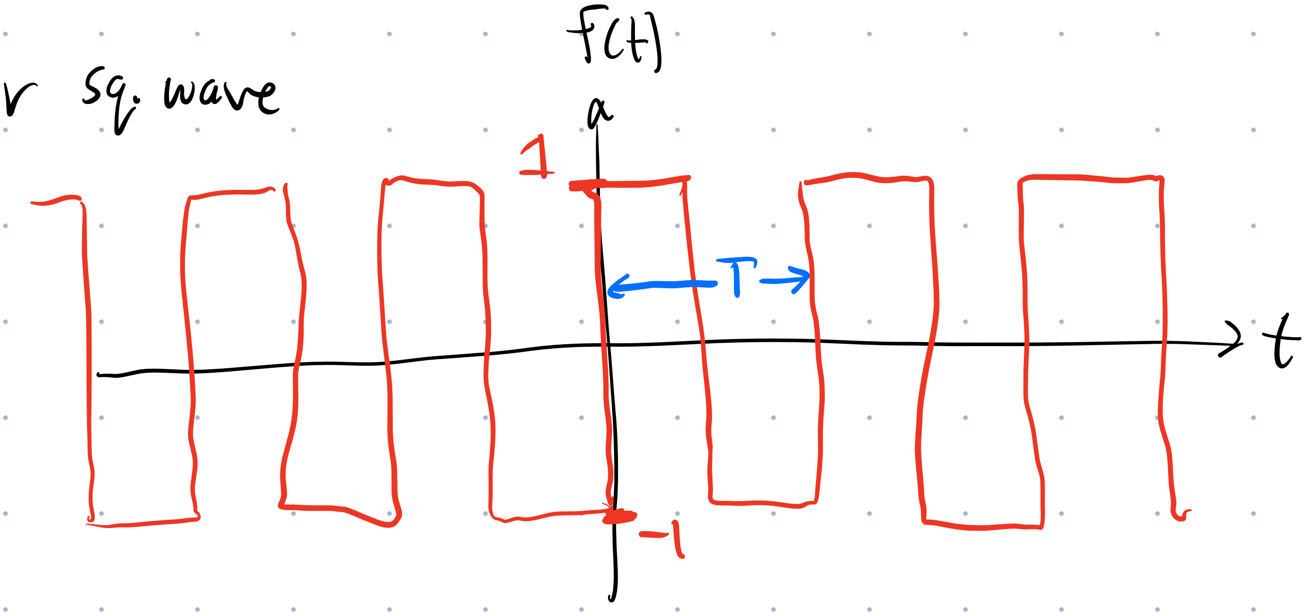
$$f(t+T) = f(t)$$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega t)\,dt$$

For sq. wave



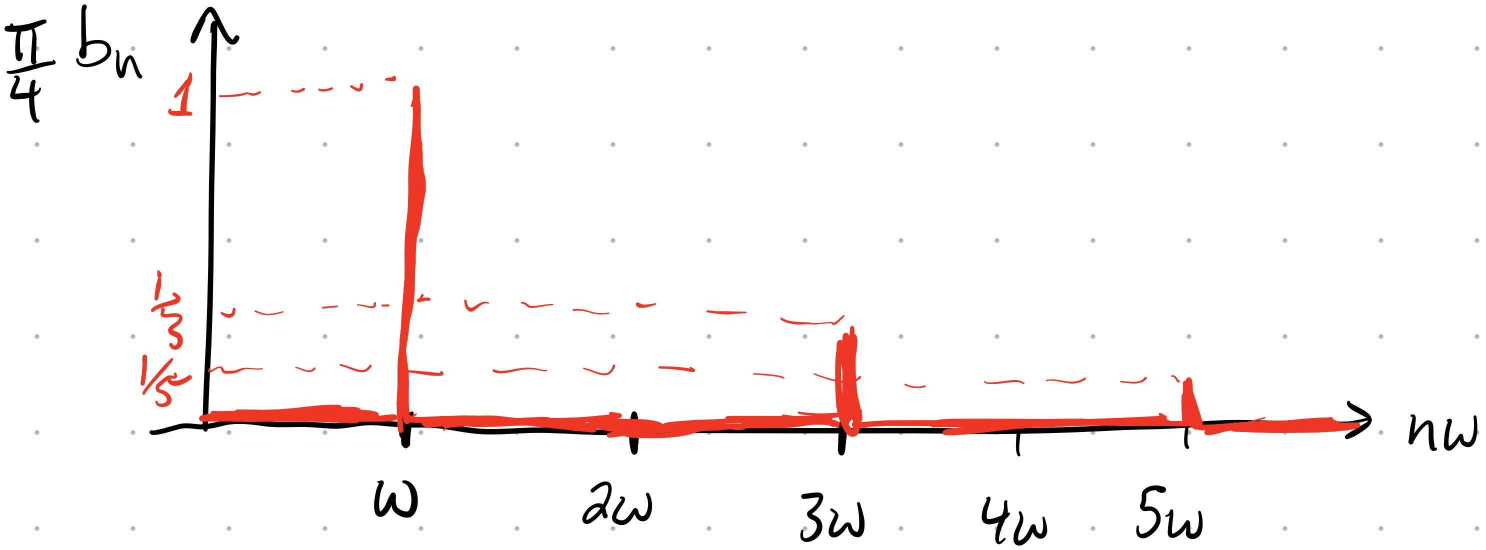
$a_n = 0 \ \forall \ n$, including $n=0$.

$b_n = 0 \ \forall \ n =$ even

$b_1 = \frac{4}{\pi} \qquad b_3 = \frac{4}{3\pi} \qquad b_5 = \frac{4}{5\pi} \ \ldots$

$b_n = \frac{4}{n\pi}$ (n odd)

$$f(t) = \frac{4}{\pi}\left[\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right]$$



Postulate that we can also express Fourier series in terms of complex exponentials.

$$\Rightarrow \quad e^{\pm jn\omega t} = \cos n\omega t \pm j\sin n\omega t$$

periodic

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$\sin n\omega t = \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}$$

$$\cos n\omega t = \frac{e^{jn\omega t} + e^{-jn\omega t}}{2}$$

If the expression above is true, need to find the C_n coeffrients.

$$\therefore f(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega t} + \sum_{n=-\infty}^{-1} C_n e^{jn\omega t}$$

$$n \rightarrow -n$$

$$\sum_{n=\infty}^{1} c_{-n} e^{-jn\omega t}$$

$$= \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega t}$$

$$f(t) = C_0 + \sum_{n=1}^{\infty} \left(C_n \underbrace{e^{jn\omega t}} + C_{-n} \underbrace{e^{-jn\omega t}} \right)$$

use Euler's Eq'n to express as sines & cosines.

$$f(t) = C_0 + \sum_{n=1}^{\infty} \Big[C_n \left(\underline{\cos n\omega t} + j \underline{\sin n\omega t} \right)$$

$$+ C_{-n} \left(\underline{\cos n\omega t} - j \underline{\sin n\omega t} \right) \Big]$$

$$= \underline{C_0} + \sum_{n=1}^{\infty} \left[\underline{(C_n + C_{-n})} \cos n\omega t + \underline{j(C_n - C_{-n})} \sin n\omega t \right]$$

The expression above is of the same form as

$$f(t) = \underline{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left[\underline{a_n}\cos n\omega t + \underline{b_n}\sin n\omega t\right]$$

$$\therefore \boxed{c_0 = \frac{a_0}{2}} \quad \boxed{c_n + c_{-n} = a_n} \quad \boxed{j(c_n - c_{-n}) = b_n}$$

mult. by $-j$

$$\boxed{c_n - c_{-n} = -jb_n}$$

add

subtract

$$2c_n = a_n - jb_n$$

$$\boxed{c_n = \frac{a_n - jb_n}{2}}$$

$$2c_{-n} = a_n + jb_n$$

$$c_{-n} = \frac{a_n + jb_n}{2}$$

$$\therefore c_n = \frac{1}{T}\left[\int_{-T/2}^{T/2} f(t)\cos n\omega t\, dt - j\int_{-T/2}^{T/2} f(t)\sin n\omega t\, dt\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\underbrace{\left[\cos n\omega t - j\sin n\omega t\right]}_{e^{-jn\omega t}} dt$$

$$\boxed{\therefore \quad C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega t}\, dt}$$

Exercise for the student, show that this expression is valid ∀ n (including $n=0$ and $n<0$).

For the square that we examined above, the exponential form of the Fourier series is:

$$C_n = \frac{\cancelto{0}{a_n} - jb_n}{2} = \frac{-jjb_n}{j\,2} = \frac{b_n}{2j}$$

$$C_{-n} = \frac{\cancelto{0}{a_n} + j b_n}{2} = \frac{j\, j b_n}{j\, 2} = -\frac{b_n}{2j} = -C_n$$

$$C_{\pm 1} = \pm \frac{b_1}{2j} = \pm \frac{\frac{4}{\pi}}{2j} = \pm \frac{2}{j\pi}$$

$$C_{\pm 3} = \pm \frac{2}{3j\pi}$$

$$\vdots$$

Sub these coefficients into our new Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$= \frac{2}{j\pi}\left[\cdots -\frac{1}{5}e^{-j5\omega t} - \frac{1}{3}e^{-j3\omega t} - e^{-j\omega t} + e^{j\omega t} + \frac{1}{3}e^{j3\omega t} + \frac{1}{5}e^{j5\omega t} + \cdots\right]$$

$$f(t) = \frac{2}{\cancel{j}\pi}\Big[\dots + \frac{1}{5}\underbrace{\left(e^{j5\omega t} - e^{-j5\omega t}\right)}_{2\cancel{j}\sin 5\omega t}$$

$$+ \frac{1}{3}\underbrace{\left(e^{j3\omega t} - e^{-j3\omega t}\right)}_{2\cancel{j}\sin 3\omega t} + \underbrace{\left(e^{j\omega t} - e^{-j\omega t}\right)}_{2\cancel{j}\sin \omega t}\Big]$$

$$\therefore f(t) = \frac{4}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \dots\right]$$

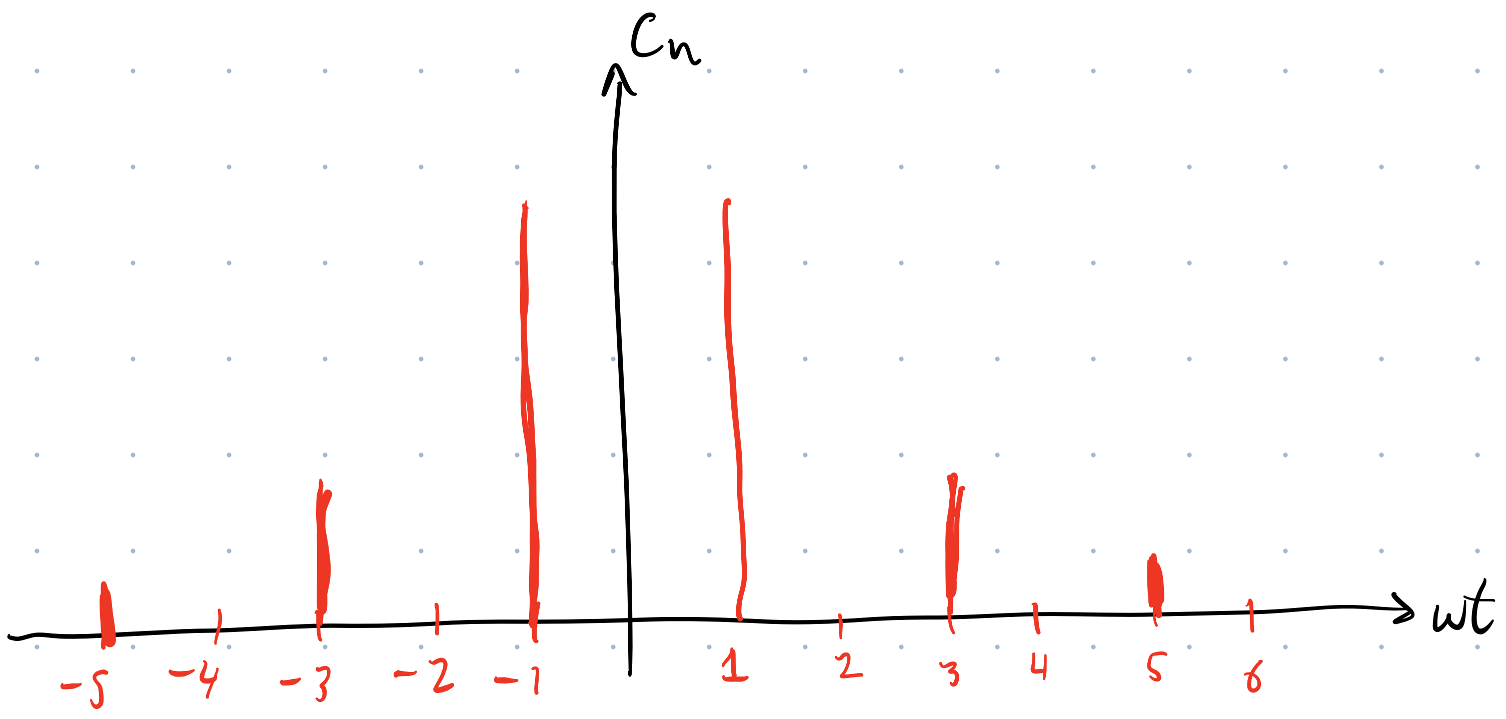
Same as before

Summary: Fourier series for periodic fns in terms of complex exponentials

$$\boxed{\begin{gathered} f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \\ c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega t}\, dt \end{gathered}}$$

Observations:

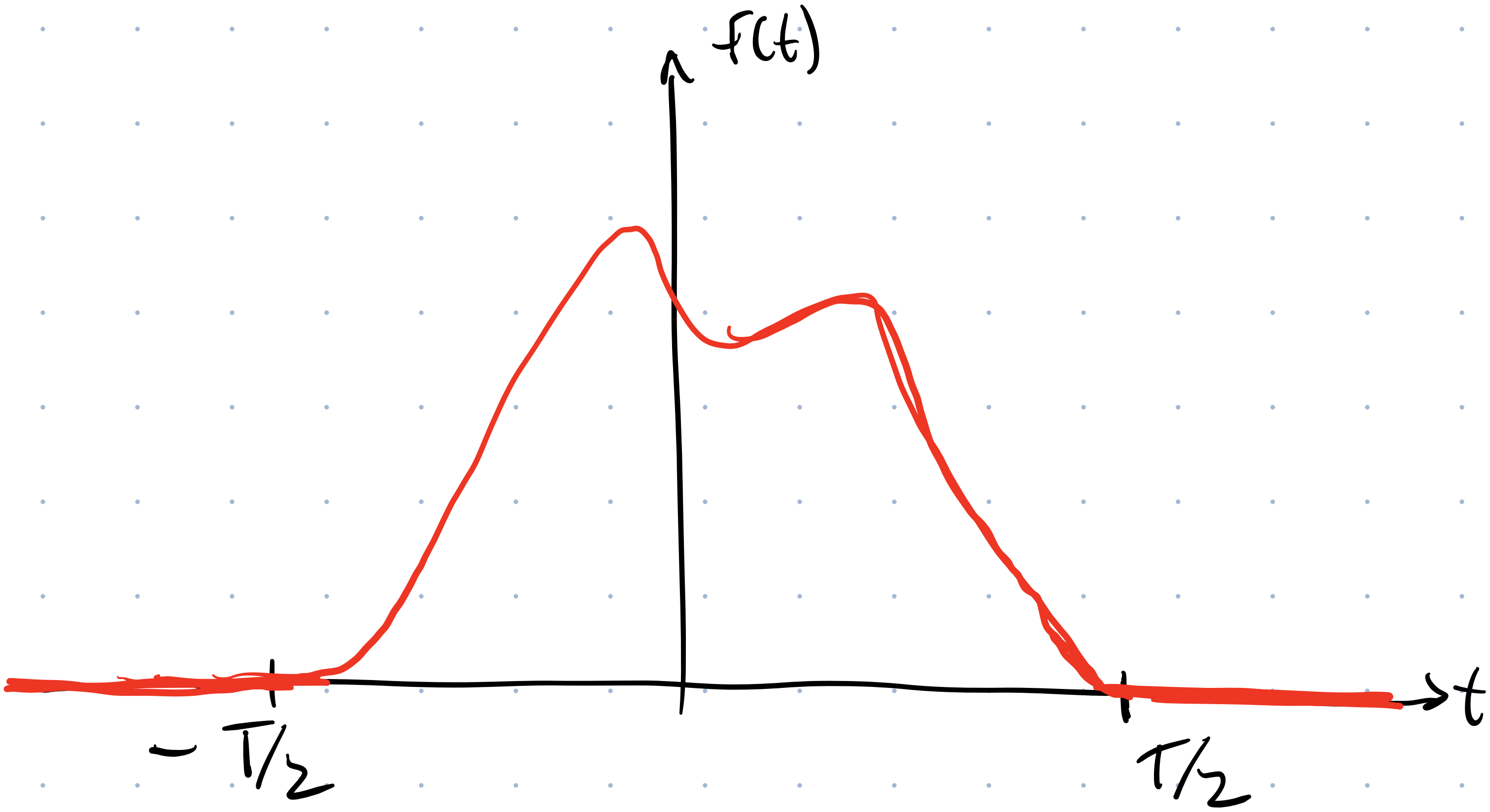
- Periodic fcns are made of discrete freq. components (spikes a certain integer values of ω & zero in between.



- Difference between neighbouring spikes decreases as T increase. In limit $T \to \infty$, expect continuous range of non-zero freq. spikes.

Want to generalize the Fourier series so that we can examine freq. content of a signal $f(t)$ that is a pulse, not periodic.

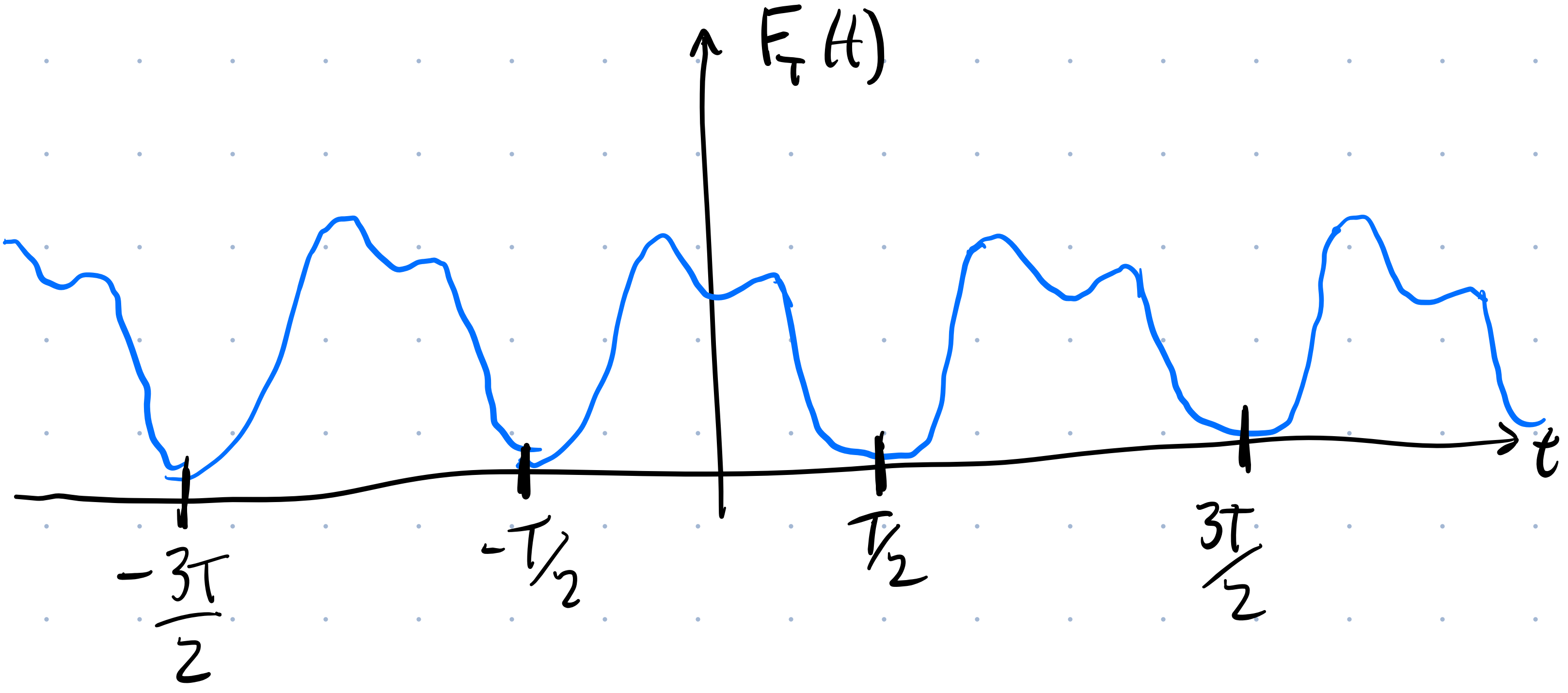
Eg. $f(t) = 0 \quad \forall \; |t| \geq \frac{T}{2}$



Now, let's construct a periodic fcn using this $f(t)$ pulse.

$$F_T(t) = f(t) \quad \text{on} \quad -\frac{T}{2} < t < \frac{T}{2}$$

our constructed periodic fcn



Thus, for $\boxed{-T/2 < t < T/2}$

⊛ $$f(t) = F_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t}$$

$$\left(\omega = \frac{2\pi}{T}\right)$$

periodic $F_T(t)$ can be written as a Fourier series

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} F_T(t) e^{-jn\frac{2\pi}{T}t}$$

Step 1: Replace $F_T(t)$ w/ $f(t)$ in C_n integral (valid b/c integral on $-\frac{T}{2}<t<\frac{T}{2}$)

$$\therefore C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\frac{2\pi}{T}t}\, dt$$

Step 2: Change limits of integration from $-T/2 \to -\infty$ & $T/2 \to +\infty$ in C_n expression. (Valid since $f(t)=0$ outside $-T/2<t<T/2$)

$$C_n = \frac{1}{T}\int_{-\infty}^{\infty} f(t)\, e^{-j\underbrace{n\frac{2\pi}{T}}_{\omega_n}t}\, dt$$

nth freq. component $\omega_n = \frac{2\pi n}{T}$

Spacing between adjacent freq. components is

$$\omega_{n+1} - \omega_n \equiv \Delta\omega = \frac{2\pi}{T}$$

$\therefore$ we can express the $\frac{1}{T}$ in front of C_n integral as $\frac{1}{T} = \frac{\Delta\omega}{2\pi}$

$$\therefore \quad C_n = \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega_n t}\, dt$$

to be cont'd.