PHYS 331 Oct. 3, 2024

- Assignment #2 is on course website.

Last Time:

For a transmission line driven by a harmonic signal, the voltage & current amplitudes along the length of the line are given by:

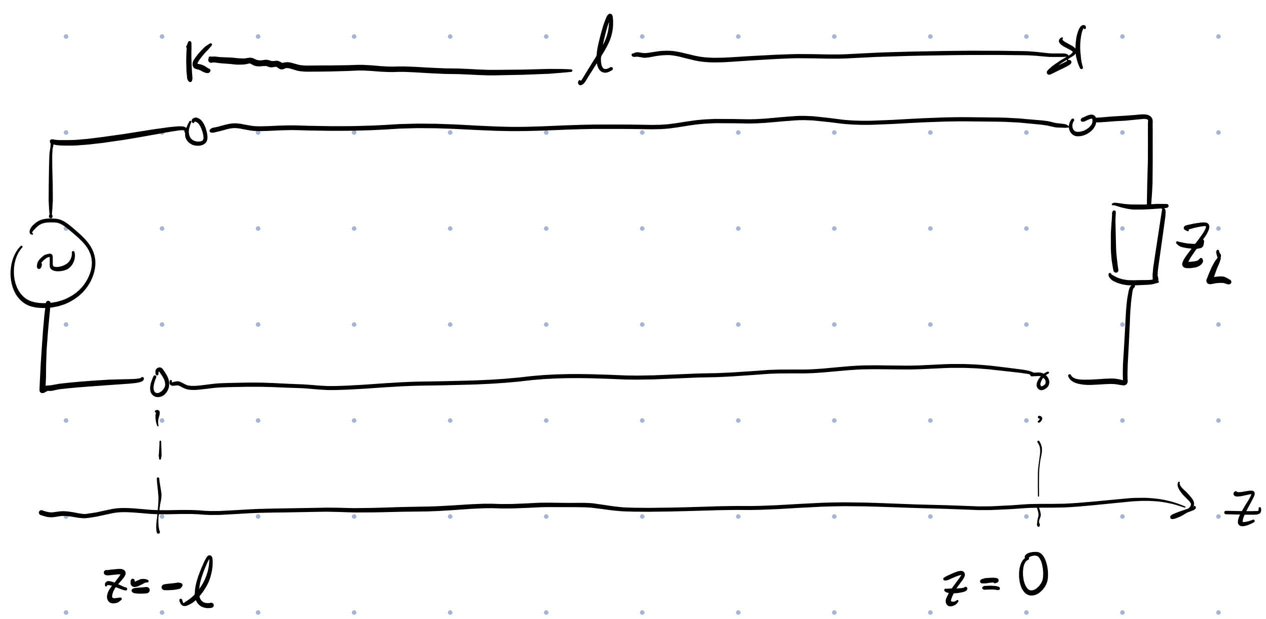
$$\boxed{\begin{aligned} V(z) &= V_+ e^{-j\beta z} + V_- e^{j\beta z} \\ I(z) &= \frac{1}{Z_0}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right] \end{aligned}}$$

$$\beta = \omega\sqrt{L_\ell C_\ell}$$ prop. const. or wavenumber

$$Z_0 = \sqrt{\frac{L_\ell}{C_\ell}}$$ characteristic impedance

In physics labs, coaxial transmission lines are designed such that $Z_0 = 50\,\Omega$.

Consider a trans. line of length l terminated by a "load" impedance Z_L at its end.



With is choice of coord. sys., the voltage & current amplitudes at the position of Z_L ($z=0$) are given by:

$$V(0) = V_+ + V_-$$
$$I(0) = \frac{1}{Z_0}\left[V_+ - V_-\right]$$

The ratio $\frac{V(0)}{I(0)}$ must be equal to the load impedance Z_L at $z=0$.

$$\frac{V(0)}{I(0)} = Z_L = \underbrace{Z_0\left[\frac{V_+ + V_-}{V_+ - V_-}\right]}_{\text{solve for } V_-.}$$

reflection coefficient Γ

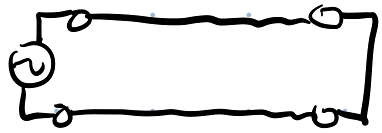
$$\text{Find}\quad V_- = V_+ \overbrace{\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)}$$

amp. of backwards travelling waves

amp. of forward travelling waves

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}} \qquad V_- = \Gamma V_+$$

Special Cases:

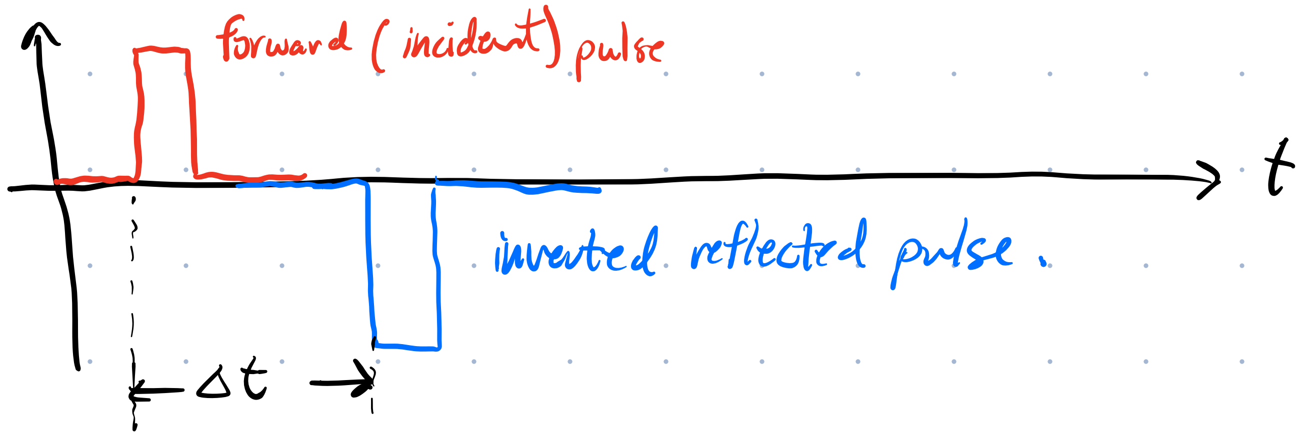


$Z_L = 0$

① $Z_L = 0$ (short circuit)

$$\Gamma = \frac{0 - Z_0}{0 + Z_0} = -1$$

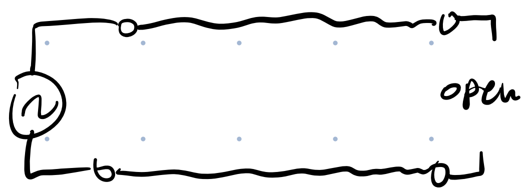
perfect reflection ($|\Gamma| = 1$), but the reflected signal is inverted relative to incident signal.



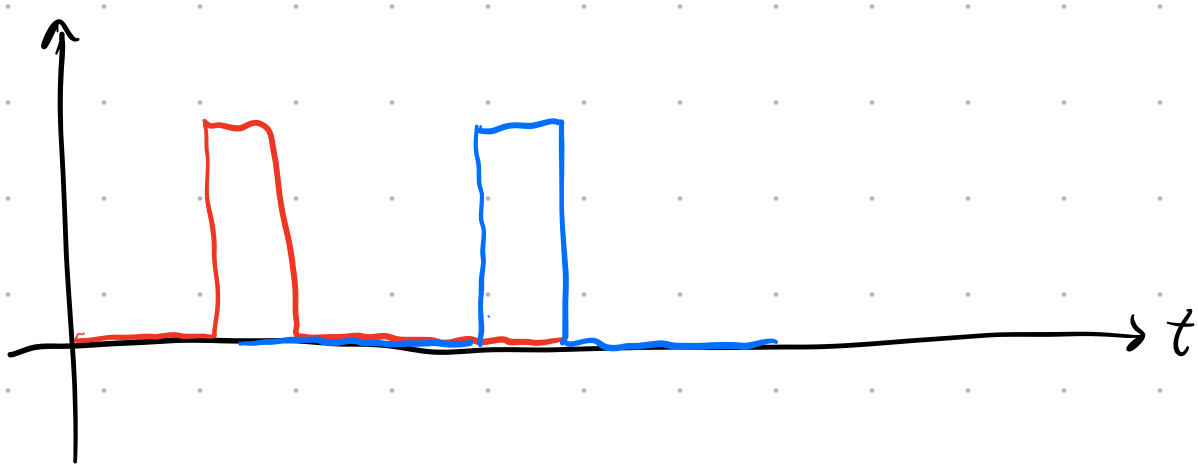
$$\Delta t = \frac{2l}{s}$$ ← propagation speed.

Time for pulse to travel length of transmission line and back again.

(2) $Z_L \to \infty$ (open circuit)



$$\Gamma = \frac{\infty - Z_0}{\infty + Z_0} = +1$$ perfect reflection w/o inversion.



③ $Z_L = Z_0$

$$\Gamma = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

no reflection. All incident signal power is absorbed by the load impedance Z_L.

This is called impedance matching.
We usually desire to acheive $Z_L = Z_0$.

When are transmission line effects important and when can they be ignored?

$$c = \lambda f \quad \therefore \quad \lambda = \frac{c}{f} = \frac{3\times10^8 \text{ m/s}}{10\times10^3 \text{ 1/s}}$$

$$= 3\times10^4 \text{ m}$$

$$= 30 \text{ km!}$$

We will find that transmission effects are important we the wavelength of the signal is comparable or shorter than the length l of the trans. line.

$f = 3\text{GHz}$

$$\lambda = \frac{3\times10^8 \text{ m/s}}{3\times10^9 \text{ 1/s}}$$

$$= 0.1 \text{ m} = 10 \text{ cm}$$

$$V(z) \approx V_+ e^{-j\beta z} + V_- e^{j\beta z}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{L_\ell C_\ell}$$

The coaxial cables used in the lab are called RG-58 & designed s.t.

$$\left.\begin{aligned} C_\ell &= 80\,\text{pF/m} \\ L_\ell &= 0.20\,\mu\text{H/m} \end{aligned}\right\} \quad Z_0 = \sqrt{\frac{L_\ell}{C_\ell}} = 50\,\Omega$$

$\beta = \omega\sqrt{L_\ell C_\ell}$ let's take as an example $\omega = 2\pi(10\text{kHz})$

$$= 2.5\times10^{-4}\ \text{m}^{-1}$$

If we take $\ell \approx 1\text{m}$

$$\beta z = 2.5\times10^{-4} \ll 1$$

$$V(z) = V_+ \cancelto{1}{e^{-j\beta z}} + V_- \cancelto{1}{e^{j\beta z}}$$

$$V(z) \approx V_+ + V_- \quad (\text{const}).$$

likewise, for current, would find $I(z) \approx \frac{1}{Z_0}\left[V_+ - V_-\right]$ (const).

When $\beta z \ll 1$, current & volt. along transmission are constant and we can safely ignore trans. line effects.

Recall $\beta = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\beta}$

using $\beta = 2.5 \times 10^{-4}\, m^{-1}$, we get $\lambda = 25km$!

When λ much longer than trans. line length l, ignore these effects.

If, on the other hand, we take

$$\omega = 2\pi(3GHz)$$

$$\beta = \omega\sqrt{L_\ell C_\ell} \approx 75\, m^{-1}$$

In this case, for $z \approx 1m$, the product βz is <u>not</u> small.

Cannot approx $e^{\pm j\beta z}$ as 1.

In fact, $e^{\pm j\beta z} = \cos\beta z \pm j\sin\beta z$

We will instead find that the voltage & current osc. along the length of transmission line.

Impedance at an arbitrary position z along the transmission line.

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$$

$$I(z) = \frac{1}{Z_0}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$$

at an arbitrary pos. z, the impedance is given by:

$$\underset{\text{impedance}}{Z}(\underset{\text{position}}{z}) = \frac{V(z)}{I(z)} = Z_0 \frac{V_+ e^{-j\beta z} + V_- e^{j\beta z}}{V_+ e^{-j\beta z} - V_- e^{j\beta z}}$$

Recall that $V_- = \Gamma V_+$

$$Z(z) = Z_0 \frac{\cancel{V_+}\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)}{\cancel{V_+}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)}$$

sub in $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)e^{j\beta z}}{e^{-j\beta z} - \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)e^{j\beta z}}$$

mult. by $\dfrac{Z_L + Z_0}{Z_L + Z_0}$

$$= Z_0 \frac{(Z_L + Z_0)e^{-j\beta z} + (Z_L - Z_0)e^{j\beta z}}{(Z_L + Z_0)e^{-j\beta z} - (Z_L - Z_0)e^{j\beta z}}$$

collect like terms of Z_L & Z_0

$$Z(z) = Z_0 \frac{Z_L\left(e^{-j\beta z} + e^{j\beta z}\right) + Z_0\left(e^{-j\beta z} - e^{j\beta z}\right)}{Z_L\left(e^{-j\beta z} - e^{j\beta z}\right) + Z_0\left(e^{-j\beta z} + e^{j\beta z}\right)}$$

use the identities $e^{j\beta z} + e^{-j\beta z} = 2\cos\beta z$

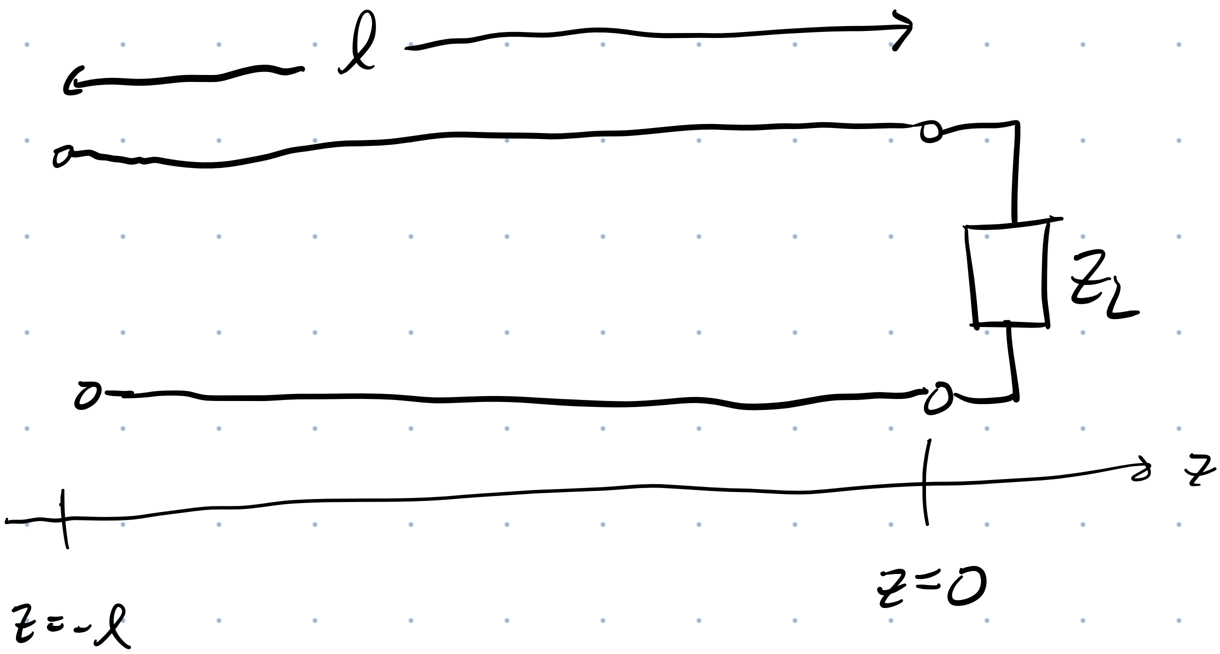
$e^{j\beta z} - e^{-j\beta z} = 2j\sin\beta z$

$$\therefore Z(z) = Z_0 \frac{\cancel{2} Z_L \cos\beta z - \cancel{2} j Z_0 \sin\beta z}{-\cancel{2} j Z_L \sin\beta z + \cancel{2} Z_0 \cos\beta z}$$

mult. by $\dfrac{\frac{1}{\cos\beta z}}{\frac{1}{\cos\beta z}}$

$$\boxed{\therefore\ Z(z) = Z_0 \frac{Z_L - jZ_0 \tan\beta z}{Z_0 - jZ_L \tan\beta z}}$$

Impedance at any pos. along trans. line that has been terminated by Z_L.



To find the impedance looking into trans. line from $z=-l$, sub in $z=-l$.

Recall that $\tan(-x) = -\tan x$

$$\Rightarrow \tan(-\beta l) = -\tan\beta l$$

Transmission line input impedance

$$Z_{in} = Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

Special cases

① $\beta l = n\pi$

Select l s.t. we satisfy $\beta l = n\pi$ where n is integer

$$\therefore \frac{2\cancel{\pi}}{\lambda} l = n\cancel{\pi} \Rightarrow l = n\left(\frac{\lambda}{2}\right)$$

Equivalent to selecting l to be an integer multiple of half wave lengths.

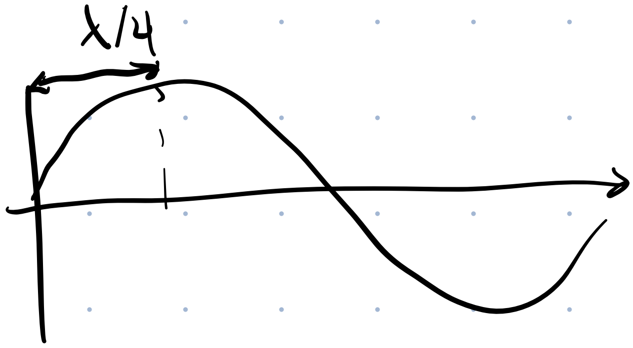
$$\Rightarrow \tan \beta l = \tan n\pi = 0.$$

$$Z_{in} = \cancel{Z_0} \frac{Z_L}{\cancel{Z_0}} = Z_L.$$

As if transmission line is not there (as if Z_L is connected directly to signal source).

(2) $\beta l = n'\left(\frac{\pi}{2}\right)$ where n' is an odd integer

$$\frac{2\cancel{\pi}}{\lambda} l = n' \frac{\cancel{\pi}}{2} \Rightarrow l = n'\left(\frac{\lambda}{4}\right)$$



length is an odd mult. of quarter wavelengths.

$$\tan \beta l = \tan\left(n' \frac{\pi}{2}\right) \rightarrow \pm\infty$$

In this case, $Z_{in} \approx Z_0 \frac{\cancel{j} Z_0 \cancel{\tan} \beta l}{\cancel{j} Z_L \cancel{\tan} \beta l} = \frac{Z_0^2}{Z_L}$

In effect, we're transforming impedances from Z_L to $\frac{Z_0^2}{Z_L}$.

(3) $Z_L = Z_0$ (impedance matching condition).

$$Z_{in} = Z_0 \frac{\cancel{Z_0 + jZ_0 \tan \beta l}}{\cancel{Z_0 + jZ_0 \tan \beta l}}$$

$$\boxed{Z_{in} = Z_0}$$

As if Z_0 has been connected directly to signal source. Usually the desired situation.

Note that, in this case, $Z_{in} = Z_0$ regardless of the length l of the trans. line.