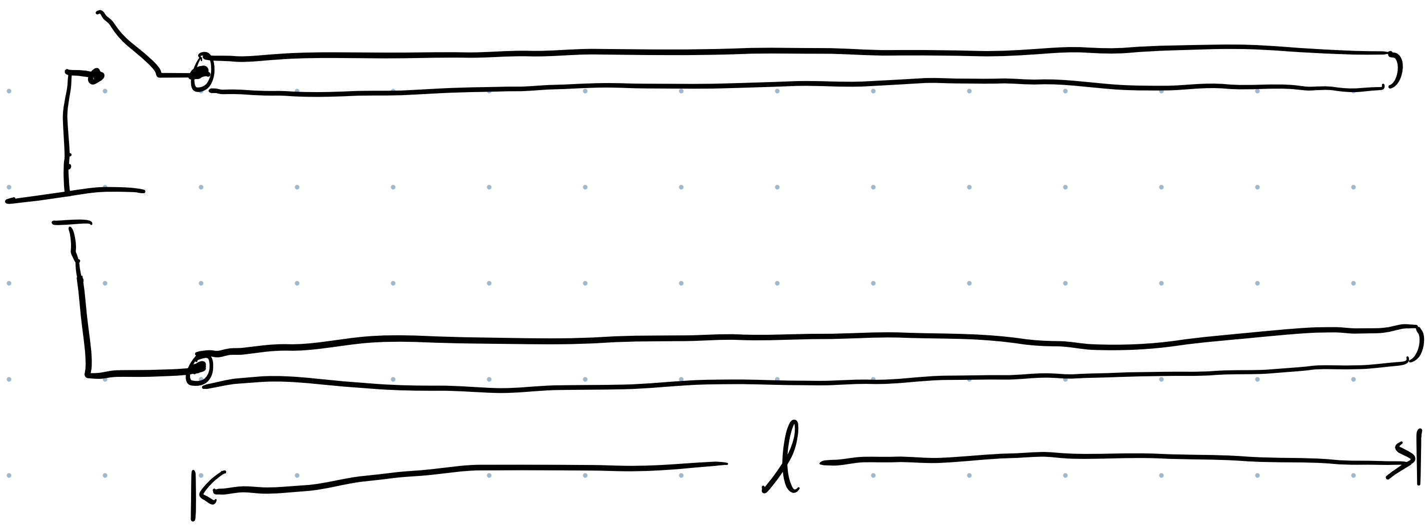
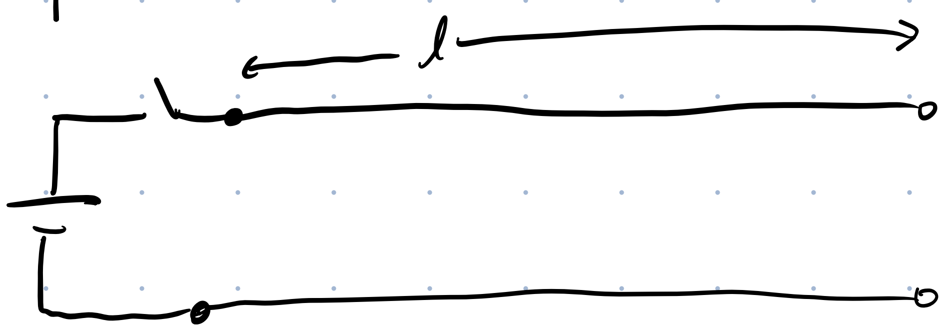
PHYS 331 Sept. 26, 2024

Last Time: Transmission lines.

Pair of parallel conductors. All transmission lines have a capacitance per unit length C_ℓ ($[C_\ell] = F/m$) & an inductance per unit length L_ℓ ($[L_\ell] = H/m$)



Model parallel wires as LC circuit.



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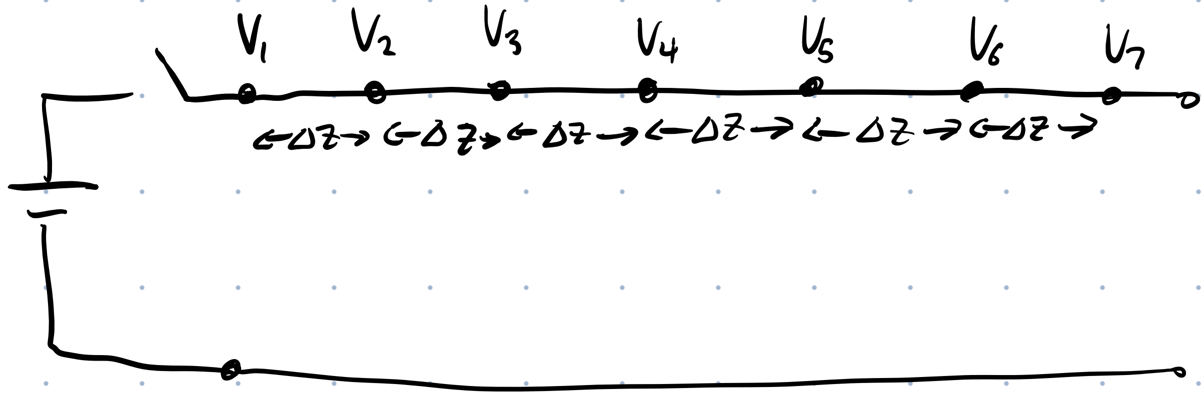
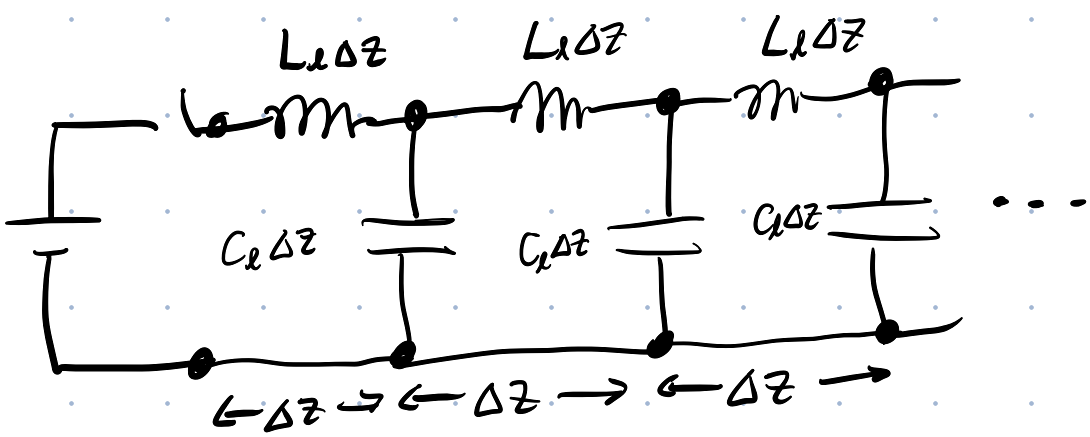
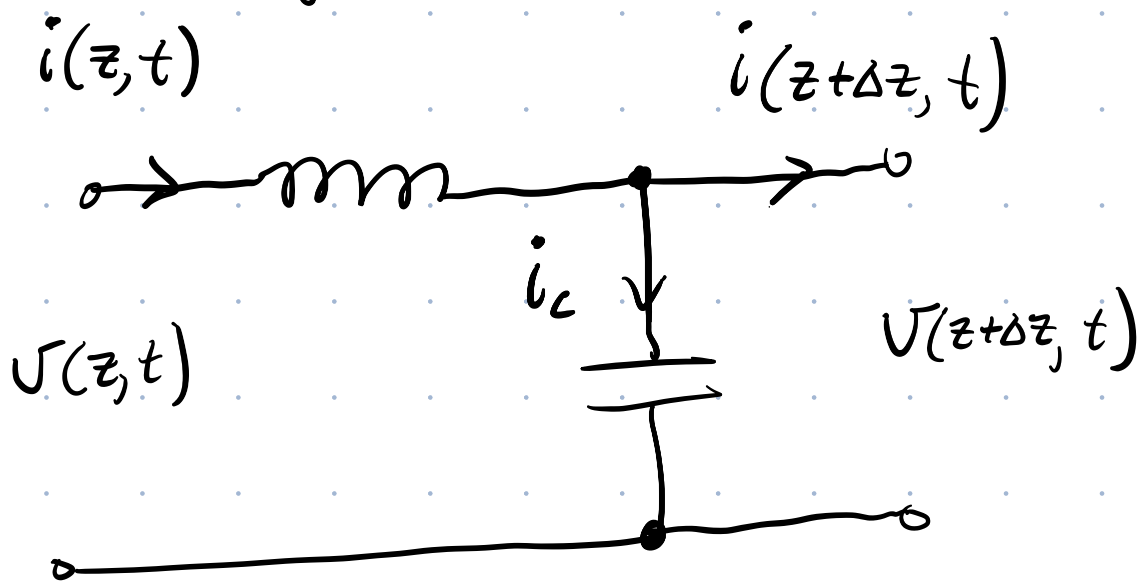


Expect $V_1 > V_2 > V_3 \cdots > V_7$ a short time after switch is closed. Simple LC circuit model fails for long trans. lines b/c can only have one volt. diff. across L.

Model long trans. line using a distributed LC network.



Consider a single branch of this circuit:



Kirchhoff Loop Rule:

$$v(z,t) - L_\ell \Delta z \frac{\partial i(z,t)}{\partial t} = v(z+\Delta z, t)$$

$$\therefore \quad \frac{v(z+\Delta z, t) - v(z,t)}{\Delta z} = -L_\ell \frac{\partial i(z,t)}{\partial t}$$

In limit $\Delta z \to 0$, we have

$$\boxed{\frac{\partial v(z,t)}{\partial z} = -L_\ell \frac{\partial i(z,t)}{\partial t}} \quad 1$$

Kirchhoff Jcn Rule:

$v(z+\Delta z, t)$

$$i(z,t) = i_c + i(z+\Delta z, t)$$

$$C = \frac{q}{v_c} \Rightarrow q = C v_c \Rightarrow i_c = \frac{dq}{dt} = C\frac{dv_c}{dt}$$

$C_\ell \Delta z$

$$i(z,t) = C_\ell \Delta z \frac{\partial}{\partial t} v(z+\Delta z, t) + i(z+\Delta z, t)$$

$$\therefore \frac{i(z+\Delta z, t) - i(z,t)}{\Delta z} = -C_\ell \frac{\partial}{\partial t} v(z+\Delta z, t)$$

limit $\Delta z \to 0$: $$\boxed{\frac{\partial i(z,t)}{\partial z} = -C_\ell \frac{\partial v(z,t)}{\partial t}}$$ 2

To make further progress, we will assume that v & i are harmonic (sinusoidal) signals.

$$v(z,t) = V(z)\, e^{j\omega t}$$
$$i(z,t) = I(z)\, e^{j\omega t}$$

position-dependent amplitudes.

From ①

$$\frac{\partial}{\partial z}\left[V(z)e^{j\omega t}\right] = -L_\ell \frac{\partial}{\partial t}\left[I(z)e^{j\omega t}\right]$$

$$\frac{\partial V(z)}{\partial z}\cancel{e^{j\omega t}} = -j\omega L_\ell I(z)\cancel{e^{j\omega t}}$$

\# $$\boxed{\therefore \frac{dV(z)}{dz} = \boxed{-j\omega L_\ell I(z)}}$$

Let's take second derivative of V w.r.t. z

$$\frac{d^2 V(z)}{dz^2} = -j\omega L_\ell \boxed{\frac{dI(z)}{dz}}$$

From ②

$$\frac{\partial}{\partial z}\left[I(z)e^{j\omega t}\right] = -C_\ell \frac{\partial}{\partial t}\left[V(z)e^{j\omega t}\right]$$

$$\therefore \frac{dI(z)}{dz} = \boxed{-j\omega C_\ell V(z)}$$

$$\therefore \frac{d^2 I(z)}{dz^2} = -j\omega C_\ell \boxed{\frac{dV(z)}{dz}}$$

$$\boxed{\begin{aligned} &\therefore \frac{d^2 V(z)}{dz^2} = -\omega^2 L_\ell C_\ell V(z) \\ &\therefore \frac{d^2 I(z)}{dz^2} = -\omega^2 L_\ell C_\ell I(z) \end{aligned}}$$

The Wave Eq'ns

General sol'n to voltage eq'n is:

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$$

Check:

$$\boxed{\frac{dV(z)}{dz} = -j\beta V_+ e^{-j\beta z} + j\beta V_- e^{j\beta z}}$$

$$\frac{d^2V(z)}{dz^2} = -\beta^2 V_+ e^{-j\beta z} - \beta^2 V_- e^{j\beta z}$$

$$= -\beta^2 \underbrace{\left[V_+ e^{-j\beta z} + V_- e^{j\beta z} \right]}_{V(z)}$$

$$\therefore \frac{d^2V(z)}{dz^2} = -\beta^2 V(z)$$

This is a solution to the wave eq'n for $V(z)$ provided

$$\beta^2 = \omega^2 L_\ell C_\ell \Rightarrow \boxed{\beta = \omega\sqrt{L_\ell C_\ell}}$$

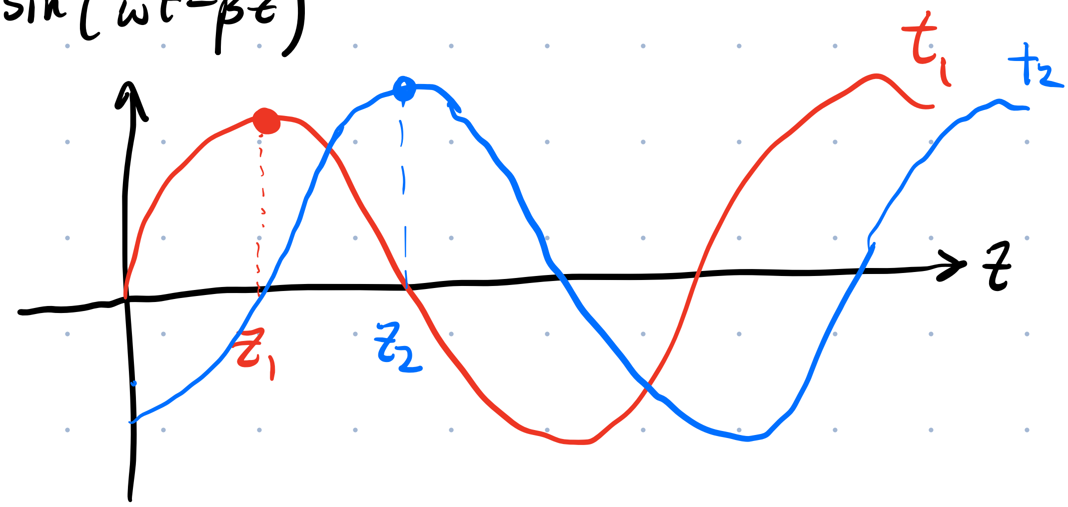
Full time-dependent voltage is $\therefore$:

$$v(z,t) = V(z)\, e^{j\omega t}$$

$$= \left[V_+ e^{-j\beta z} + V_- e^{j\beta z} \right] e^{j\omega t}$$

$$\therefore\ v(z,t) = V_+ e^{j(\omega t - \beta z)} + V_- e^{j(\omega t + \beta z)}$$

Consider $\sin(\omega t - \beta z)$



Require: $\sin(\omega t_1 - \beta z_1) = \sin(\omega t_2 - \beta z_2)$

$$\therefore\ \omega t_1 - \beta z_1 = \omega t_2 - \beta z_2$$

$$\omega \underbrace{(t_2 - t_1)}_{\Delta t} = \beta \underbrace{(z_2 - z_1)}_{\Delta z}$$

$$\therefore\ \frac{\omega}{\beta} = \underbrace{\frac{\Delta z}{\Delta t}}_{S \text{ the speed of the signal.}}$$

$S = \dfrac{\omega}{\beta}$ know $\beta = \omega\sqrt{L_\ell C_\ell}$

$$s = \frac{\cancel{\omega}\,1}{\cancel{\omega}\sqrt{L_\ell C_\ell}}$$

$$\boxed{s = \frac{1}{\sqrt{L_\ell C_\ell}}}$$

Last class, for a parallel pair of conductors of radius a separated by dist. d, we had:

$$C_\ell = \frac{\pi \varepsilon_0}{\ln\left(\frac{d}{a}\right)} \qquad L_\ell = \frac{\mu_0}{\pi} \ln\left(\frac{d}{a}\right)$$

$$\therefore L_\ell C_\ell = \mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$$\boxed{\therefore s = c}$$

$\therefore$ our signal on the transmission line travels at the speed of light.

In general, if our parallel wires were surrounded by a dielectric and/or a material w/ a relative permeability μ_r, then $C_\ell \to \varepsilon_r C_\ell$

$$L_\ell \to \mu_r L_\ell$$

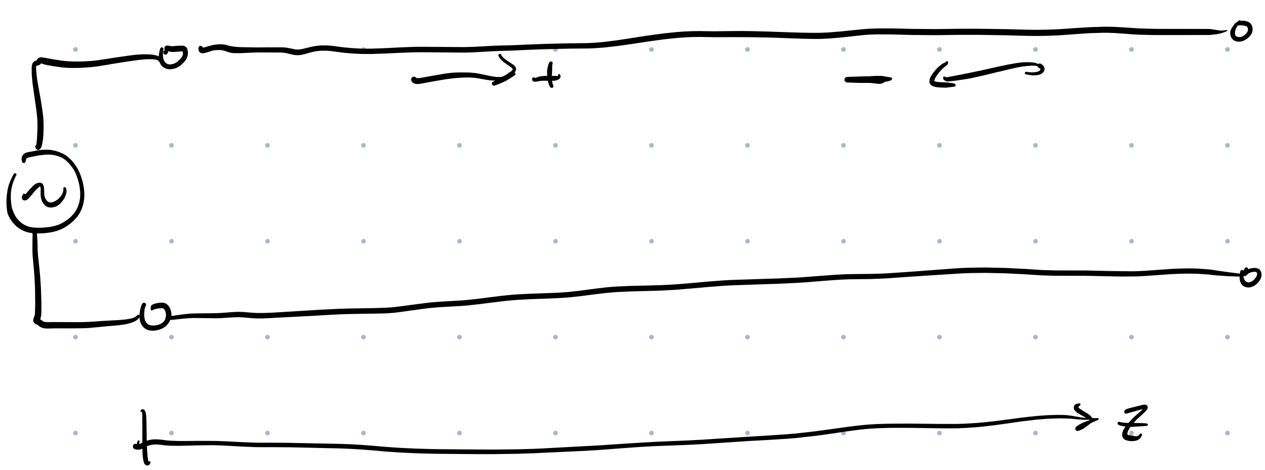
Then S would become

$$S = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{n}$$ where n is refractive index.

If we analyzed the $\omega t + \beta z$ term in $V(z,t)$ sol'n, we would find the same result for S, but w/ opposite sign ; $S = -\frac{1}{\sqrt{L_\ell C_\ell}} = -c$.

$V_+ e^{j(\omega t - \beta z)}$: forward travelling wave.

$V_- e^{j(\omega t + \beta z)}$; backwards travelling wave.



Backwards travelling waves are due to reflections from the opposite end of the transmission.

What is the physical interpretation of β?

$$S = \frac{\omega}{\beta} = \lambda f$$

$$\Downarrow$$

$$\frac{2\pi \cancel{f}}{\beta} = \lambda \cancel{f} \Rightarrow \boxed{\beta = \frac{2\pi}{\lambda}}$$

Called wave number or propagation constant.

Return to eq'ns (#) & (*). Both expressions for $\frac{dV(z)}{dz}$ & must be equal.

$$-\cancel{j}\beta V_+ e^{-j\beta z} + \cancel{j}\beta V_- e^{j\beta z} = -\cancel{j}\omega L_\ell I(z)$$

Solve for $I(z)$:

$$I(z) = \frac{\beta\left[-V_+ e^{-j\beta z} + V_- e^{j\beta z}\right]}{-\omega L_\ell}$$

$$\therefore \quad I(z) = \frac{\cancel{\omega}\sqrt{L_\ell C_\ell}}{\cancel{\omega} L_\ell}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$$

$$\therefore \quad \underset{\text{units A}}{I(z)} = \underbrace{\sqrt{\frac{C_\ell}{L_\ell}}}_{\text{must have units of } \frac{1}{\Omega}.}\underbrace{\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]}_{\text{units V}}$$

Define $\sqrt{\frac{L_\ell}{C_\ell}}$ as the "characteristic impedance" of the transmission line.

$$\boxed{Z_0 \equiv \sqrt{\frac{L_\ell}{C_\ell}}}$$