PHYS 331 Sept. 12, 2024

Review of complex nos. & their physical meaning.

Recall that we can express any complex number as:

① mag & phase

$$\underline{z} = |z| e^{j\phi} = |z|(\overbrace{\cos\phi + j\sin\phi}^{e^{j\phi}\ (\text{Euler})})$$

② real & imaginary components

$$\underline{z} = \underbrace{x}_{\text{real part}} + j\underbrace{y}_{\text{imaginary part.}}$$

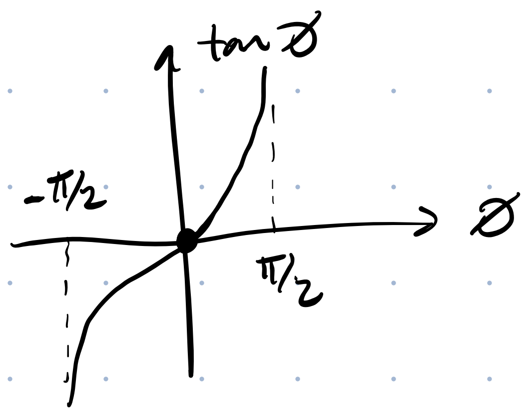
Switch between the two representations:

$$\therefore \underline{x} + j\underline{y} = |z|(\underline{\cos\phi} + j\,\underline{\sin\phi})$$

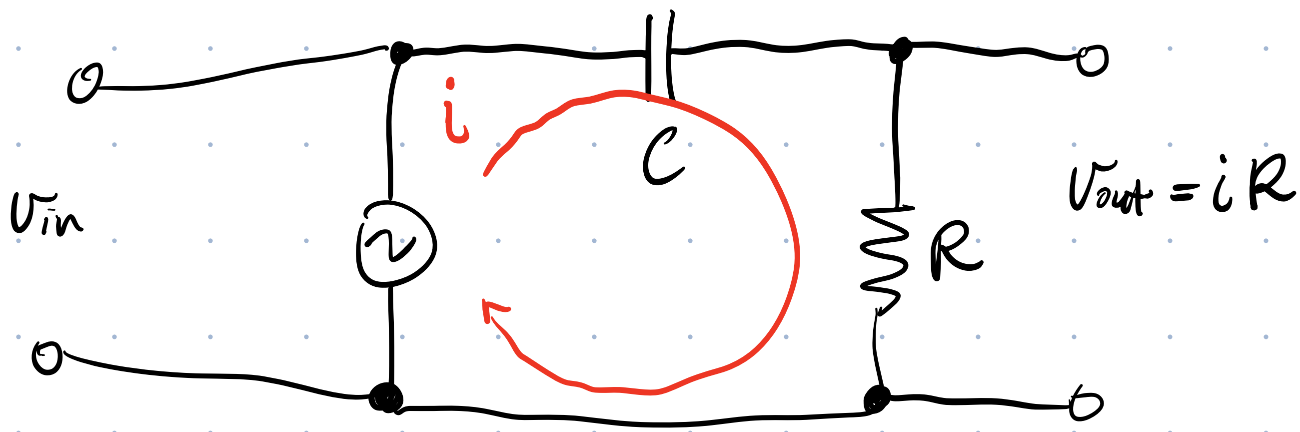
$$\boxed{\begin{aligned} x &= |z|\cos\phi \\ y &= |z|\sin\phi \\ \therefore\ |z| &= \sqrt{x^2+y^2} \\ \tan\phi &= \frac{y}{x} \end{aligned}}$$

If $\mathrm{Im}[z] = y = 0$, then $\tan\phi = 0$

$\therefore\ \phi = 0.$



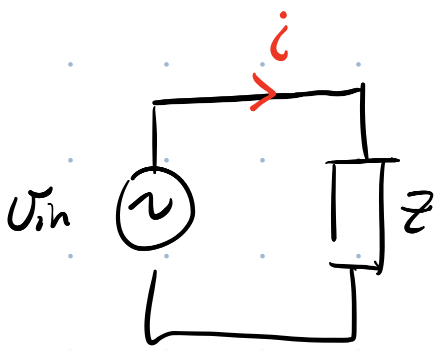
Consider a example from electronics.
RC series circuit.



$v_{in} = V_{in} e^{j\omega t}$ $\qquad$ $v_{out} = V_{out} e^{j(\omega t + \phi)}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad = iR$

To fully determine v_{out}, need to find the ampl. V_{out} & phase ϕ.



$$i = \frac{v_{in}}{Z}$$

$$Z = \frac{1}{j\omega C} + R = \frac{1 + j\omega RC}{j\omega C}$$

$$i = v_{in} \left(\frac{j\omega C}{1 + j\omega RC} \right) \frac{1 - j\omega RC}{1 - j\omega RC}$$

$$= v_{in} \left(\frac{j\omega C + \omega^2 R C^2}{1 + (\omega RC)^2} \right)$$

$$i = v_{in} \, \omega C \, \frac{(j + \omega RC)}{1 + (\omega RC)^2}$$

$$\therefore\ v_{out} = iR = v_{in}\,\frac{\omega RC(j+\omega RC)}{1+(\omega RC)^2}$$

Recall $v_{in} = V_{in}e^{j\omega t}$

$v_{out} = V_{out}\,e^{j(\omega t+\phi)} = V_{out}\,e^{j\omega t}e^{j\phi}$

$$V_{out}\cancel{e^{j\omega t}}\,\underbrace{e^{j\phi}}_{\cos\phi + j\sin\phi} = V_{in}\cancel{e^{j\omega t}}\,\frac{\omega RC(j+\omega RC)}{1+(\omega RC)^2}$$

$$V_{out}(\cos\phi + j\sin\phi) = V_{in}\,\frac{\omega RC(j+\omega RC)}{1+(\omega RC)^2}$$

real & imaginary parts on left must equal the real & imagary parts on right, respectively.

Real: $$V_{out}\cos\phi = V_{in}\,\frac{(\omega RC)^2}{1+(\omega RC)^2} \equiv X$$

Imaginary: $V_{out} \sin\phi = V_{in} \dfrac{\omega RC}{1+(\omega RC)^2} \equiv Y$

X is the real component of V_{out}
Y " " imaginary " " "

To find the mag. of V_{out}, evaluate $\sqrt{X^2+Y^2}$

$$\sqrt{X^2+Y^2} = \sqrt{\frac{V_{in}^2 (\omega RC)^4}{(1+(\omega RC)^2)^2} + \frac{V_{in}^2 (\omega RC)^2}{(1+(\omega RC)^2)^2}}$$

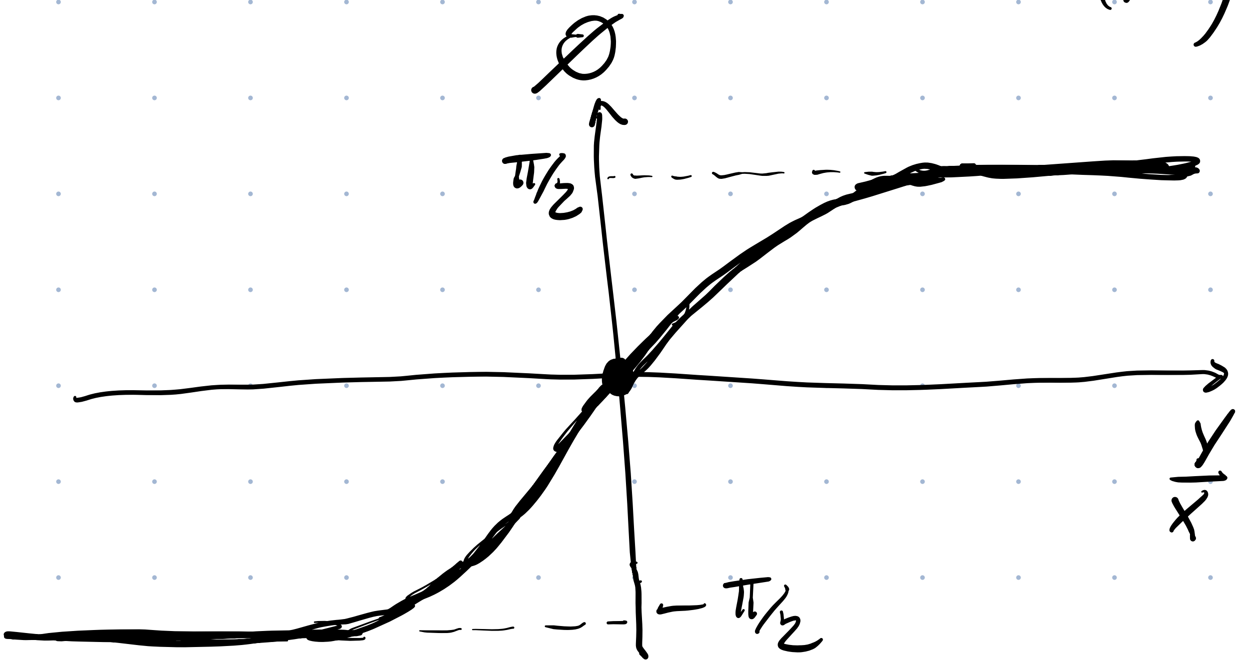
$$= \frac{V_{in}\,\omega RC}{1+(\omega RC)^2}\sqrt{(\omega RC)^2+1}$$

$$\therefore \underbrace{\sqrt{X^2+Y^2}}_{V_{out}} = \boxed{V_{in} \frac{\omega RC}{\sqrt{1+(\omega RC)^2}} = V_{out}}$$

Phase : $\tan\phi = \dfrac{Y}{X} = \dfrac{\dfrac{\cancel{V_{in}}\, \cancel{\omega RC}^{\,\to 1}}{\cancel{1+(\omega RC)^2}}}{\dfrac{\cancel{V_{in}}\, (\omega RC)^{\cancel{2}}}{\cancel{1+(\omega RC)^2}}}$

$\therefore \tan\phi = \dfrac{1}{\omega RC}$ $\quad\therefore\ \phi = \tan^{-1}\left(\dfrac{Y}{X}\right)$

$= \tan^{-1}\left(\dfrac{1}{\omega RC}\right)$



Notice: when $Y = 0$, $\phi = 0$.

$\therefore$ In this case, the V_{in} & V_{out} are in phase when V_{out} is purely real.

If $Y > 0$, the phase between V_{out} & V_{in} is non-zero & pos.

If $Y < 0$, the phase between V_{out} & V_{in} is non-zero & neg.

A lock-in detector is able to measure the real & imaginary components of small AC signals even when they are buried in noise. True provided we know the frequency of the signal that we want to detect.

If we use lock-in to meas. X & Y we can then calc. the mag. & phase of desired signal.

How does a lock-in detector meas. X & Y?

Lock-in detectors have two inputs

signal of interest $V_S = A_S \sin(\omega_S t + \phi)$

reference input $V_r = A_r \sin(\omega_r t + \theta)$

We know everything about reference V_r.

We control A_r, ω_r, θ

Want to determine A_S & ϕ of V_S.

① Internally, lock-in detector multiplies the two inputs.

$$V_{out} = V_S V_r = A_S A_r \sin(\omega_S t + \phi) \sin(\omega_r t + \theta)$$

$$= \frac{A_S A_r}{2} \left\{ \cos\left[(\omega_S t + \phi) - (\omega_r t + \theta)\right] - \cos\left[(\omega_S t + \phi) + (\omega_r t + \theta)\right] \right\}$$

low freq. term

$$\therefore V_{out} = \frac{A_s A_r}{2}\left\{ \cos\left[(\omega_s - \omega_r)t + (\phi - \theta)\right] - \cos\left[(\omega_s + \omega_r)t + (\phi + \theta)\right]\right\}$$

high freq. term.

② Next, the multiplied signal is passed through a low-pass filter to eliminate the high-freq. term.

$$V_{out} \rightarrow V_{out}' = \frac{A_s A_r}{2}\cos\left[(\omega_s - \omega_r)t + (\phi - \theta)\right]$$

(after filter)

③ The filtered signal is then averaged over a long time.

time average.

If $\omega_s \neq \omega_r$, then $\langle V_{out}' \rangle = 0$

b/c $\cos((\ )t)$ is half the time pos. & half the time neg. → averages to zero.

Lock-in detector is blind to signals w/ freq. $\omega_s \neq \omega_r$.

If $\omega_s = \omega_r$, then

$$\langle v_{out}' \rangle_{\omega_s = \omega_r} = \frac{A_s A_r}{2} \cos(\phi - \theta) \qquad (*)$$

In $(*)$, we still have two unknowns (A_s, ϕ) and only one eq'n.

④ To establish two eq'ns in the two unknowns,

(a) Set $\theta = 0$. i.e. we control reference input, arbitrarily set its phase to zero.

$$\langle v_{out}' \rangle_{\omega_s = \omega_r} \Big|_{\theta = 0} = \frac{A_s A_r}{2} \cos\phi$$

$$= \frac{A_r}{2} \Big[\underbrace{A_s \cos\phi}_{\equiv X} \Big]$$

real component of V_s, signal of interest.

$$\boxed{\therefore X = \frac{2}{A_r}\langle V_{out}'\rangle_{\omega_s=\omega_r}\Big|_{\theta=0}}$$

real component

(b) Set $\theta = \frac{\pi}{2}$

$$\langle V_{out}'\rangle_{\omega_s=\omega_r}\Big|_{\theta=\frac{\pi}{2}} = \frac{A_s A_r}{2}\underbrace{\cos\left(\phi - \frac{\pi}{2}\right)}_{\sin\phi}$$

$$= \frac{A_r}{2}\left[\underbrace{A_s \sin\phi}_{\equiv Y}\right]$$

$\equiv Y$ imaginary component of V_s

$$\boxed{Y = \frac{2}{A_r}\langle V_{out}'\rangle_{\omega_s=\omega_r}\Big|_{\theta=\pi/2}}$$

Lock-in detector is capable of meas. X & Y of our AC signal. Can then find amp. & phase of v_s using:

$$A_s = \sqrt{X^2 + Y^2} \qquad \tan\phi = \frac{Y}{X}$$