PHYS 301 Nov. 27, 2024

Last Time:

- Dielectrics in an external electric field become polarized w/ polarization $\vec{P}$

- Get "bound" charges:

  - $\rho_b = -\vec{\nabla} \cdot \vec{P}$
  - $\sigma_b = \vec{P} \cdot \hat{n}$

- Electric Displacement $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

  - $\vec{\nabla} \cdot \vec{D} = \rho_f$
  - $\oint \vec{D} \cdot d\vec{l} = Q_f$

Today: Griffiths 4.4 <u>Linear Dielectrics</u>

For electric fields that are not too strong, for many materials $\vec{P} \propto \vec{E}$.

When $\vec{P}$ is prop to $\vec{E}$, the proportionality constant is called the <u>electric susceptibility</u> <u>χ_e</u>

$$\circledast \quad \vec{P} = \varepsilon_0 \chi_e \vec{E} \qquad (\chi_e \text{ is unitless.})$$

χ_e is a material property and materials that obey $\circledast$ are called <u>linear dielectrics.</u>

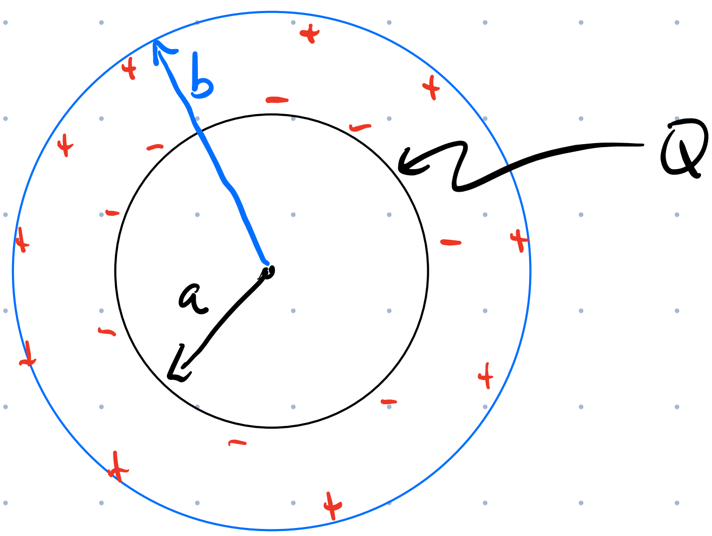
Note that $\vec{E}$ in $\circledast$ is the <u>total</u> electric field, that which causes the polarization plus the contribution from the polarized material.

$$\begin{aligned} \vec{D} &= \varepsilon_0 \vec{E} + \vec{P} \\ &= \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} \\ &= \varepsilon_0 (1 + \chi_e) \vec{E} \end{aligned}$$

For linear dielectrics, $\vec{D}$ is also prop. to $\vec{E}$.

The quantity $\varepsilon_0(1+\chi_e) \equiv \varepsilon$ is called the permittivity of the material & $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$ is the relative permittivity or dielectric constant.

Eg. A conducting sphere has charge Q & is surrounded by a linear dielectric of permittivity ε. Find the potential @ centre.



charge on conductor.

Know $\oint \vec{D} \cdot d\vec{a} = Q_f = Q$

Pick a Gaussian surface w/ radius $r > a$.

$$\oint \vec{D}\cdot d\vec{a} = D\,4\pi r^2 = Q$$

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{r}$$

$$\therefore \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \frac{Q}{4\pi r^2}\,\hat{r}$$

$$= \underbrace{\varepsilon_0 (1+\chi_e)}_{\varepsilon} \vec{E} = \frac{Q}{4\pi r^2}\,\hat{r}$$

$$\therefore \quad \vec{E} = \frac{Q}{4\pi \varepsilon r^2}\,\hat{r}$$

If $a<r<b$, $\quad \vec{E} = \dfrac{Q}{4\pi\varepsilon r^2}\,\hat{r}$

if $r>b$, then $\varepsilon = \varepsilon_0 \quad \vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\,\hat{r}$

$$\Delta V = \cancelto{0}{V(\infty)} - V(0) = -\int_0^\infty \vec{E}\cdot d\vec{l}$$

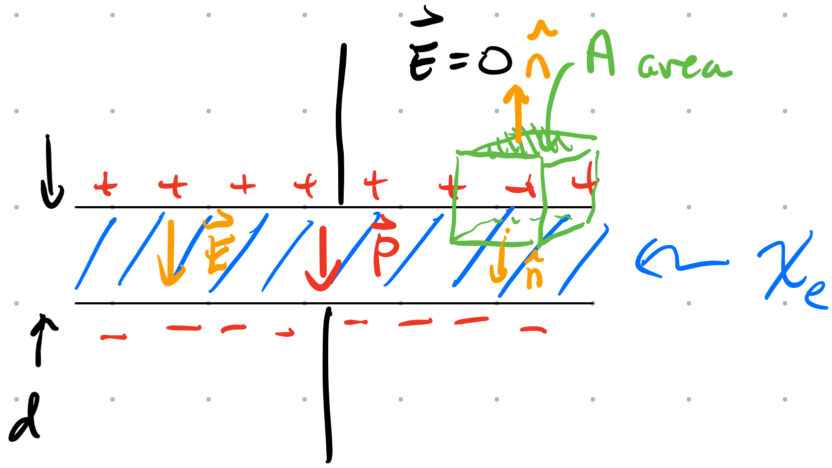
$$\therefore \quad V(0) = \int_0^\infty \vec{E}\cdot d\vec{l}$$

$$= \int_0^a \cancelto{0}{\vec{E}\cdot d\vec{l}} + \int_a^b \frac{Q}{4\pi\varepsilon r^2}\,dr \int_b^\infty \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$

$$= \frac{Q}{4\pi}\left[-\frac{1}{\varepsilon r}\Big|_a^b - \frac{1}{\varepsilon_0 r}\Big|_b^\infty \right]$$

$$\boxed{\therefore V(0) = \frac{Q}{4\pi}\left[-\frac{1}{\varepsilon b}+\frac{1}{\varepsilon a}+\frac{1}{\varepsilon_0 b}\right]}$$

Eg. Find $\vec{E}$ between the plates of a cap that has been filled w/ a linear dielectric.



$$\oint \vec{D}\cdot d\vec{a} = Q_f$$

If our cap has charge per unit area σ, then $Q_f = \sigma A$.

$$\oint \vec{D}\cdot d\vec{a} = \underbrace{-\cancelto{0}{D_{out}A}}_{\text{top}} + \underbrace{D_{in}A}_{\text{btm}} = \sigma A$$

$$D_{in} = \varepsilon E$$

$$\therefore \quad \varepsilon E \cancel{A} = \sigma \cancel{A}$$

$$\therefore \quad E = \frac{\sigma}{\varepsilon} = \frac{Q}{\varepsilon_r \varepsilon_0 A}$$

$$\therefore \Delta V = Ed = \frac{Qd}{\varepsilon_r \varepsilon_0 A}$$

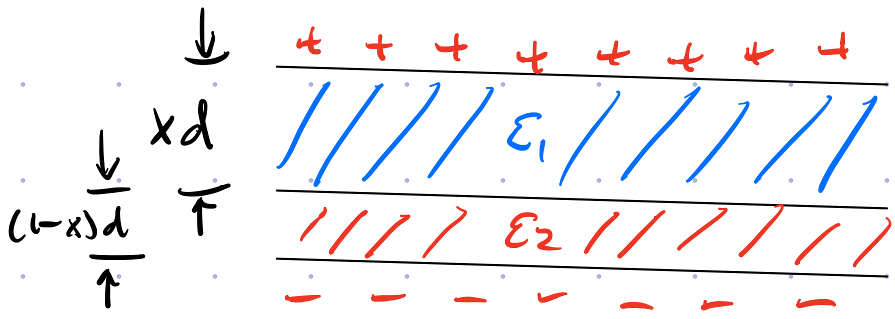
$$\therefore \frac{Q}{\Delta V} = C = \varepsilon_r \underbrace{\varepsilon_0 \frac{A}{d}}_{C_0}$$

C_0 ← cap. w/ nothing between plates.

$$\boxed{C = \varepsilon_r C_0}$$

The cap. is enhanced by a factor of the dielectric const.

Eg. Find the cap of a parallel-plate cap partially filled w/ dielectric ε_1 & partially filled w/ dielectric ε_2.



where $0 < x < 1$

From previous prob., $E_1 = \dfrac{\sigma}{\varepsilon_1}$ $\quad E_2 = \dfrac{\sigma}{\varepsilon_2}$

$$\therefore \Delta V = -\int_0^{xd} \vec{E}_1 \cdot d\vec{l} - \int_{xd}^{d} \vec{E}_2 \cdot d\vec{l}$$

$$= -\frac{\sigma}{\varepsilon_1} xd - \frac{\sigma}{\varepsilon_2} \underbrace{(d - xd)}_{(1-x)d}$$

$$\Delta V = -\frac{Q\,x\,d}{\varepsilon_{r,1}\,\varepsilon_0 A} - \frac{Q\,(1-x)\,d}{\varepsilon_{r,2}\,\varepsilon_0 A}$$

Know $C_0 = \varepsilon_0 \dfrac{A}{d}$

$$\Delta V = -\frac{Q}{C_0}\underbrace{\left[\frac{x}{\varepsilon_{r,1}} + \frac{1-x}{\varepsilon_{r,2}}\right]}_{\frac{x\varepsilon_{r,2} + (1-x)\varepsilon_{r,1}}{\varepsilon_{r,1}\varepsilon_{r,2}}}$$

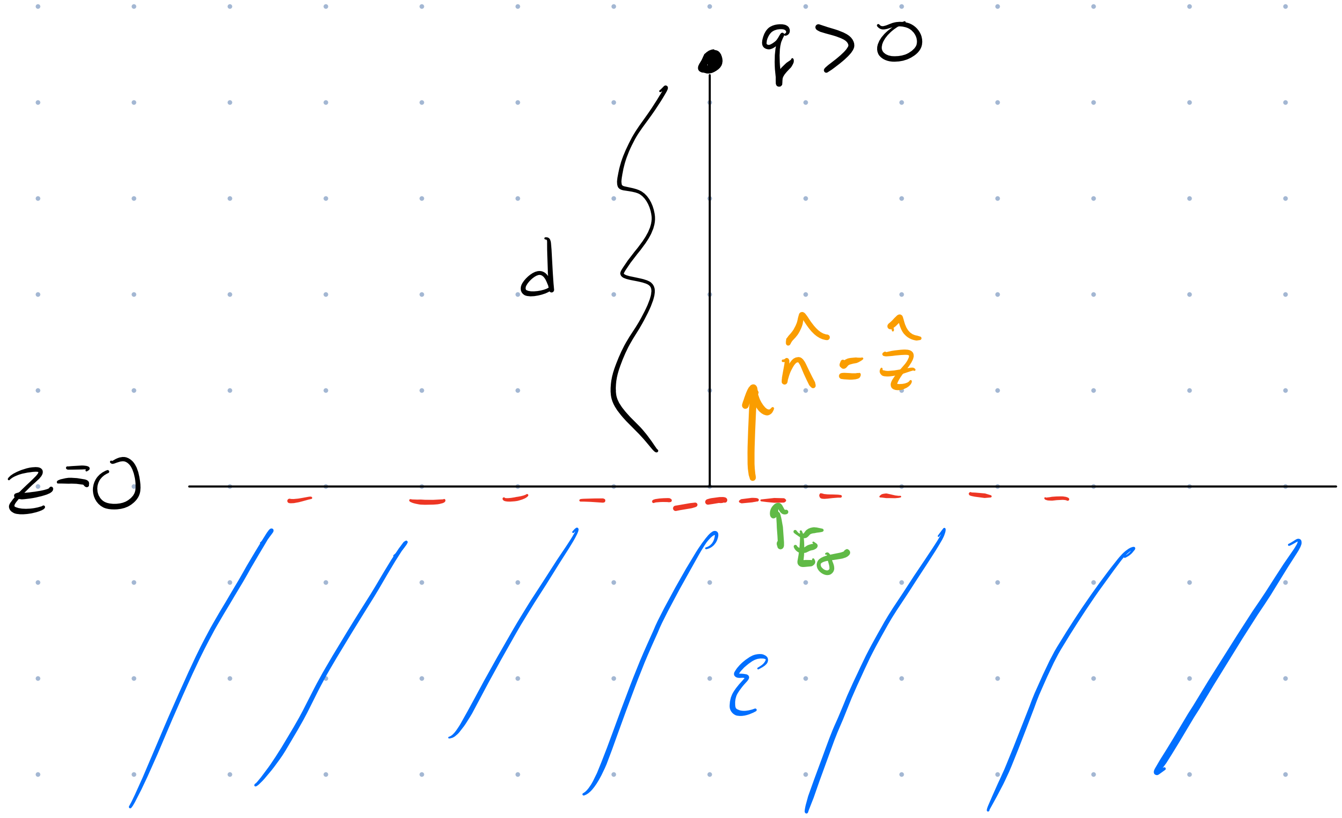
$$\boxed{C = \frac{Q}{|\Delta V|} = C_0 \frac{\varepsilon_{r,1}\,\varepsilon_{r,2}}{\varepsilon_{r,1}(1-x) + \varepsilon_{r,2}\,x}}$$

Check: ① if $x=1$, $C = \varepsilon_{r,1} C_0$ ✓

② if $x=0$, $C = \varepsilon_{r,2} C_0$ ✓

③ if $\varepsilon_{r,1} = \varepsilon_{r,2} = 1$ $C = C_0$ ✓

Eg. Find the charge density induced on a linear dielectric that fills the space $z<0$ when a pt. q is at $z=d$.



First, consider $\rho_b = -\vec{\nabla}\cdot\vec{P}$

If $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

$$\therefore \rho_b = -\varepsilon_0 \chi_e \underbrace{\vec{\nabla}\cdot\vec{E}}_{\frac{\rho_b+\rho_f}{\varepsilon_0}}$$

$$\therefore \rho_b = -\chi_e \rho_b - \chi_e \rho_f$$

$$\boxed{\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f}$$

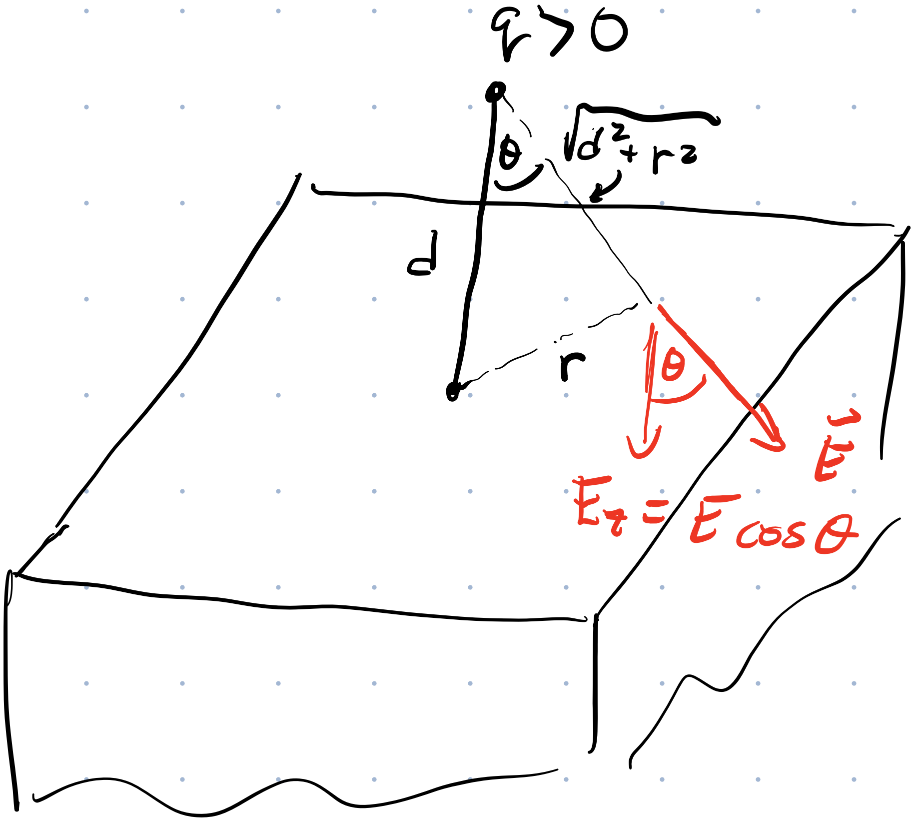
In our problem, there is no free charge in the dielectric. $\therefore\ \rho_b = 0$.

$$\sigma_b = \vec{P}\cdot\hat{n} = P_z = \varepsilon_0 \chi_e E_z(0)$$

@ $z=0$ (just inside the dielectric)

Need to find $\vec{E}$ @ some arbitrary pt @ dielectric surface. Two contrib. (one from pt. charge q & one from bound charge σ_b).

① From q.



$$\cos\theta = \frac{d}{\sqrt{r^2+d^2}}$$

$$E_{q,z} = \frac{-1}{4\pi\varepsilon_0} \frac{q}{d^2+r^2} \frac{d}{\sqrt{r^2+d^2}} = \frac{-qd}{4\pi\varepsilon_0 \left[d^2+r^2\right]^{3/2}}$$

overall pos. b/c $\sigma_b < 0$.

② From σ_b $\qquad E_\sigma = \frac{-\sigma_b}{2\varepsilon_0}$

ε_0 b/c calc. E from the bound charge itself will contain χ_e & σ_b itself will contain χ_e which is related to ε_r

$$\sigma_b = \varepsilon_0 \chi_e \left[\frac{-qd}{4\pi\varepsilon_0 \left[d^2+r^2\right]^{3/2}} - \frac{\sigma_b}{2\varepsilon_0} \right]$$

solve for σ_b

$$\sigma_b \underbrace{\left(1 + \frac{\chi_e}{2}\right)}_{\frac{2+\chi_e}{2}} = \frac{-qd\chi_e}{4\pi (d^2+r^2)^{3/2}}$$

$$\boxed{\therefore \sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{qd}{\left[d^2+r^2\right]^{3/2}}}$$

Note that

total bound charge.

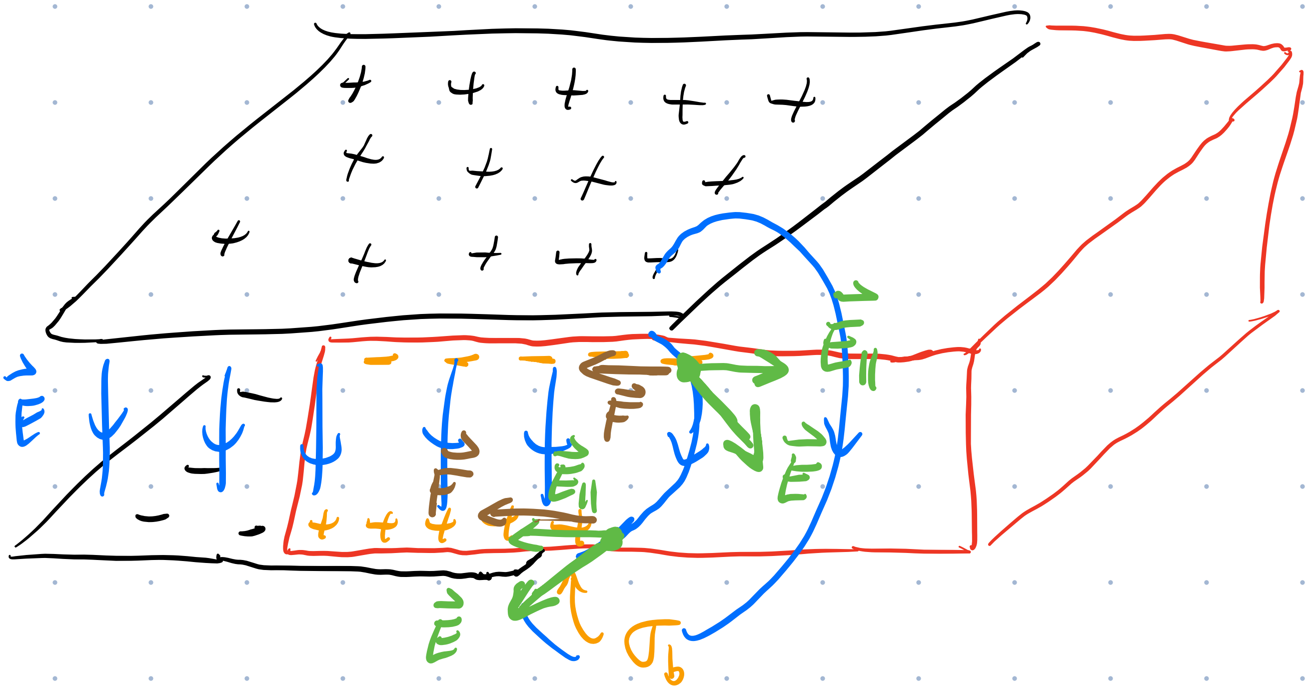
$$q_b = \int \sigma_b \, da = -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right) q d \underbrace{\int \frac{r\,dr\,d\phi}{(r^2+d^2)^{3/2}}}_{\frac{2\pi}{d}}$$

$$\boxed{q_b = -\left(\frac{\chi_e}{2+\chi_e}\right) q}$$

If $\chi_e = 0$, $q_b = 0$ ✓

If $\chi_e \to \infty$, $q_b = -q$ ✓

4.4.4 Forces on Dielectrics



Dielectric partially inside a cap.

The parallel components of the "fringing" fields push the dielectric into the cap.