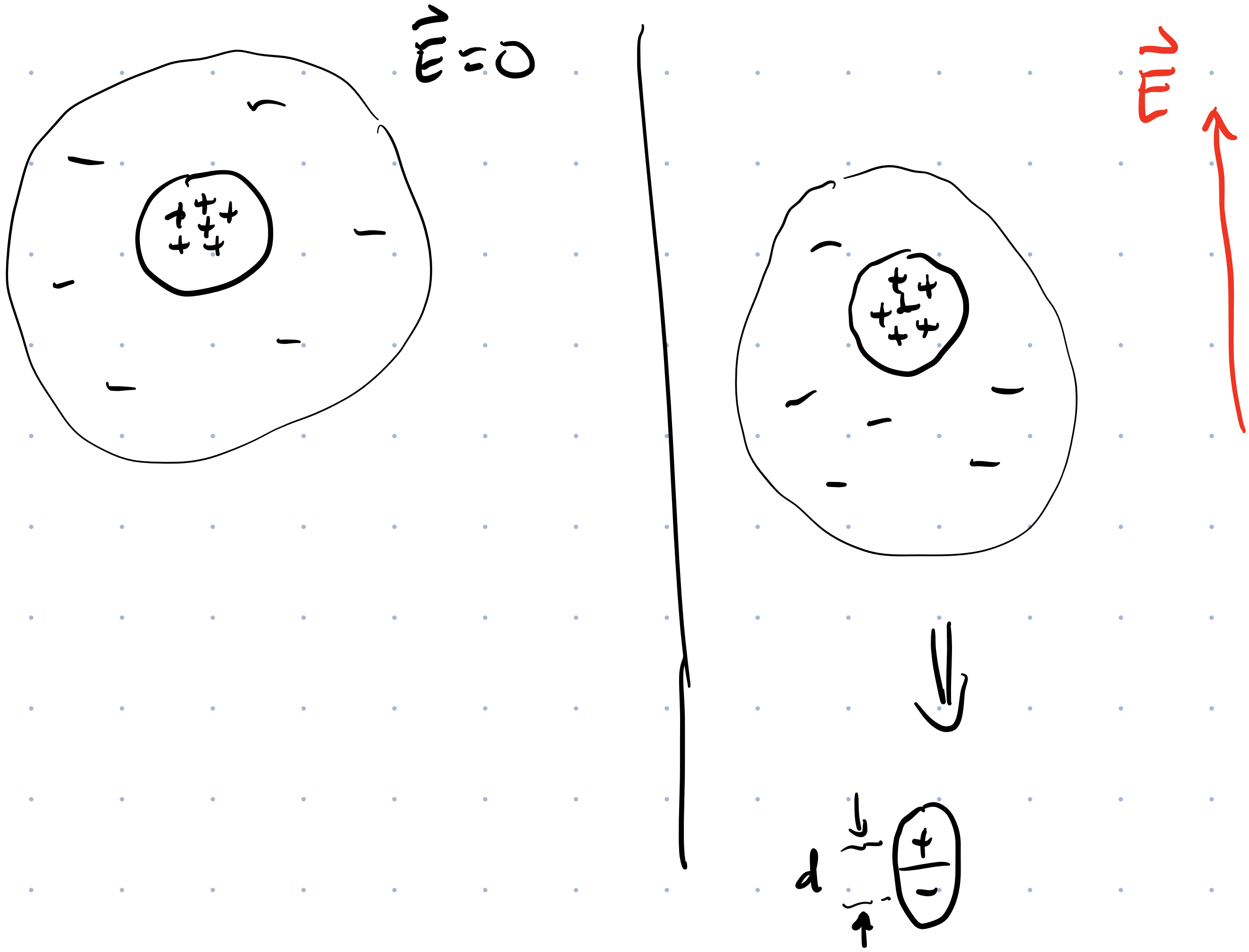
PHYS 301

Nov. 25, 2024

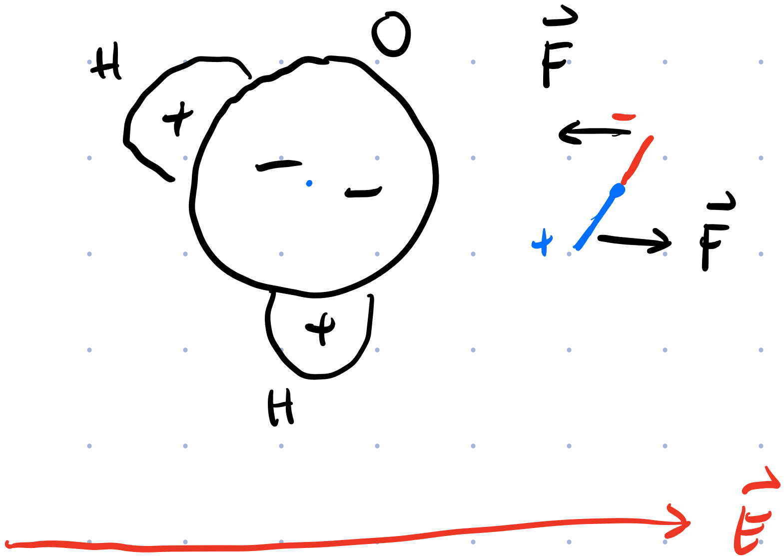
Griffiths Ch. 4 Polarization

Induced dipoles:

Neutral atom in an $\vec{E}$-field.

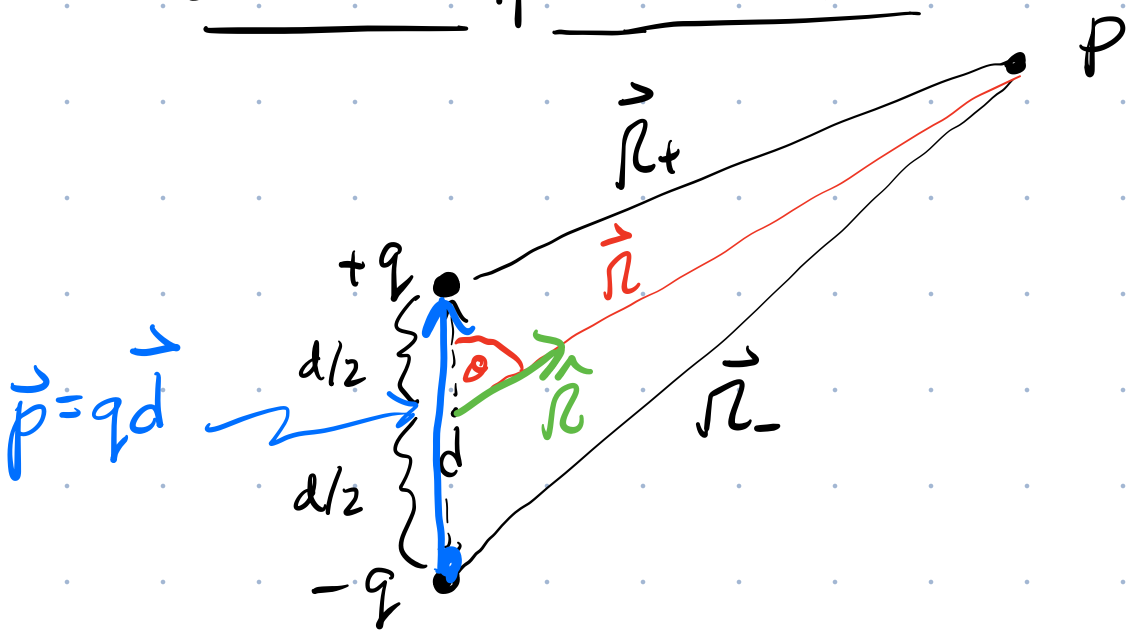


Polar molecules in a electric field



In either case (neutral atoms or polar molecules), get a separation of charge when place in an $\vec{E}$-field.

Electric Dipole Potential



$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$ Find potential @ P when $\mathscr{r} >> d$, far field.

$$\mathscr{r}_+ = \sqrt{\mathscr{r}^2 + \left(\frac{d}{2}\right)^2 - d\mathscr{r}\cos\theta}$$

$$= \mathscr{r}\sqrt{1 + \left(\frac{d}{2\mathscr{r}}\right)^2 - \frac{d}{\mathscr{r}}\cos\theta}$$

If $\frac{d}{\mathscr{r}} << 1$

$$\frac{1}{\mathscr{r}_+} \approx \frac{1}{\mathscr{r}}\left(1 - \frac{d}{\mathscr{r}}\cos\theta\right)^{-1/2}$$

use binomial approx $(1+x)^p \approx 1 + px$ when $|x| << 1$

$$\frac{1}{\mathscr{r}_+} \approx \frac{1}{\mathscr{r}}\left(1 + \frac{d}{2\mathscr{r}}\cos\theta\right)$$

Likewise $\frac{1}{\mathscr{r}_-} = \frac{1}{\mathscr{r}}\left(1 - \frac{d}{2\mathscr{r}}\cos\theta\right)$

$$\therefore V = \frac{q}{4\pi\varepsilon_0 r}\left[\left(\cancel{1} + \frac{d}{2r}\cos\theta\right) - \left(\cancel{1} - \frac{d}{2r}\cos\theta\right)\right]$$

$$V \approx \frac{q\, d\cos\theta}{4\pi\varepsilon_0 r^2}$$

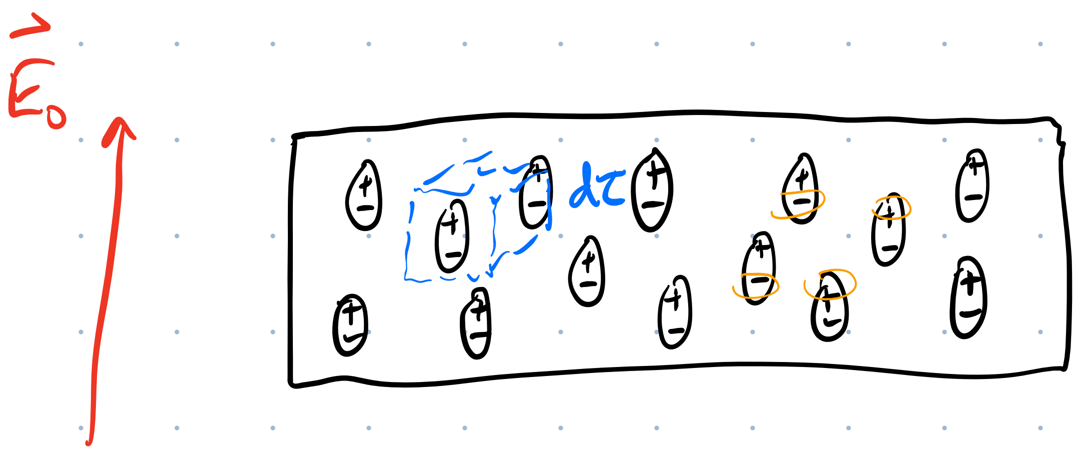
If $\vec{p} = q\vec{d}$

then $\vec{p}\cdot\hat{r} = qd\cos\theta$

$$V = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

$\vec{p}$ is the dipole moment

In a insulator or dielectric in an electric field, get a large collection of small dipoles that collectively can have a large effect.



The material becomes polarized & the polarization is:

$\vec{P}$ = dipole moment ($\vec{p}$) per unit volume.

The polarized object will create its own electric field. To calc. $\vec{E}$, start w/ potential V.

For a single dipole, know $V = \dfrac{\vec{p} \cdot \hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2}$

For a volume element $d\tau'$ inside material

$$\vec{p} = \vec{P}(\vec{r}\,')\, d\tau'$$

$$\therefore V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{P(\vec{r}\,') \cdot \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' \quad \#$$

Want to re-express (#) in a more revealing form:

Note that $\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

x-compent:

$$\frac{\partial}{\partial x'}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= \frac{-\frac{1}{2}(-2)(x-x')}{\left[(\quad)^2+(\quad)^2+(\quad)^2\right]^{3/2}}$$

$$= \frac{(x-x')}{\mathscr{r}^3}$$

with all 3 componants, find

$$\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\therefore$ we can write # as

$$V = \frac{1}{4\pi\varepsilon_0} \int_V \vec{P} \cdot \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

Product Rule (5)

$$\vec{P} \cdot \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \vec{\nabla}' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\left(\vec{\nabla}' \cdot \vec{P}\right)$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\int_V \vec{\nabla}' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) d\tau'}_{\text{apply div. th.}} - \int_V \frac{1}{\mathscr{r}}\left(\vec{\nabla}' \cdot \vec{P}\right) d\tau'\right]$$

$$= \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\vec{P}}{\mathscr{r}} \cdot d\vec{a}' - \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{\nabla}' \cdot \vec{P}}{\mathscr{r}} d\tau'$$

c. t.

$$\frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma}{\mathscr{r}}\, da'$$

c. t.

$$\frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{\mathscr{r}}\, d\tau'$$

$d\vec{a} = da\,\hat{n}$

$\therefore\ \vec{P}\cdot\hat{n} \equiv \sigma_b$

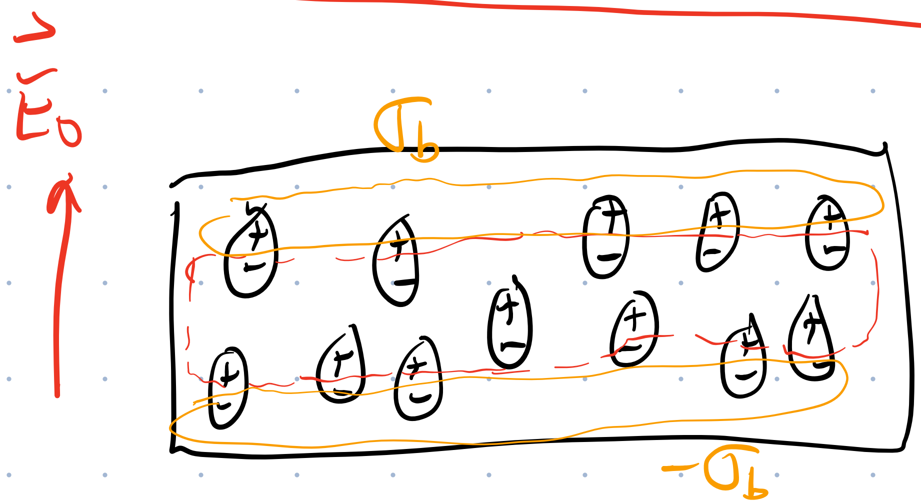
bound surface charge.

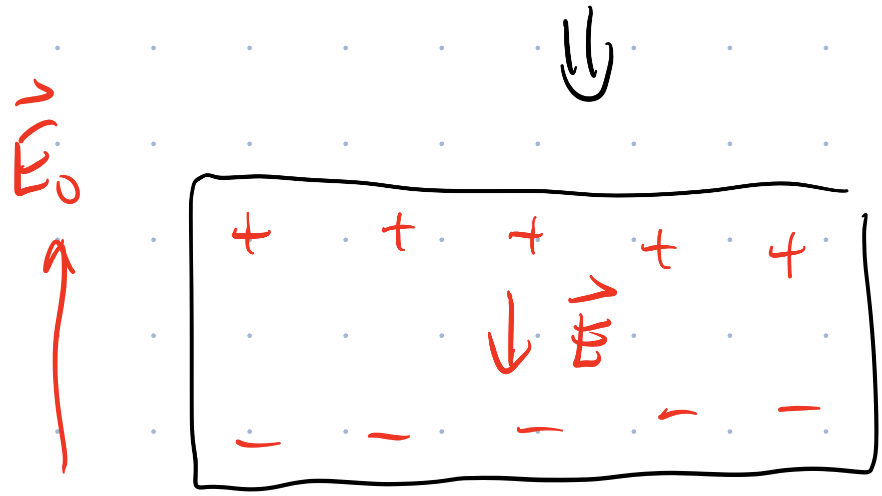
$\therefore\ -\vec{\nabla}\cdot\vec{P} \equiv \rho_b$ (drop primes)

bound volume charge density.

non-zero only when $\vec{P}$ is <u>not</u> uniform.

$$\boxed{\therefore\ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_b}{\mathscr{r}}\, d\tau'}$$

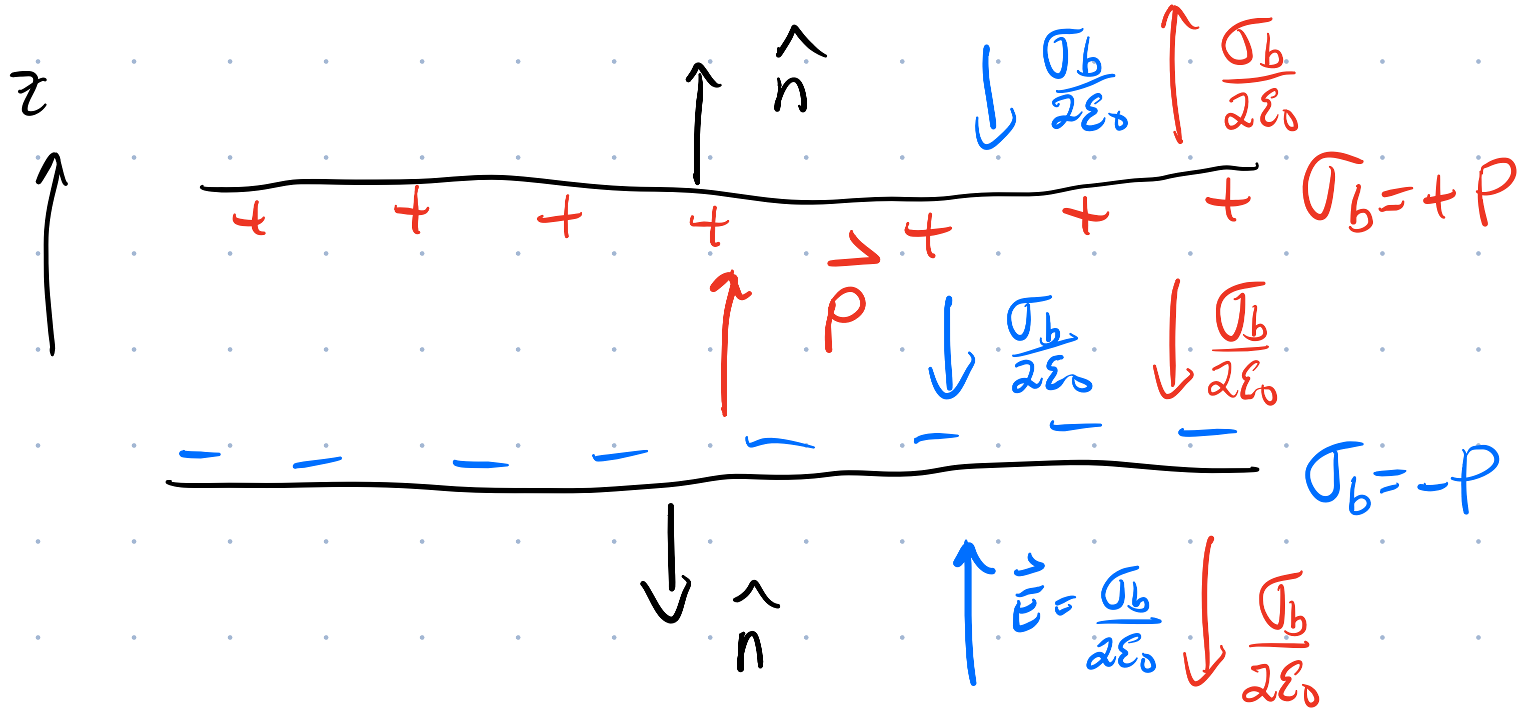




Eg. Find $\vec{E}$ due to a uniformly-polarized slab.

Since $\vec{P}$ is given as uniform,

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0.$$



top: $\sigma_b = \vec{P} \cdot \hat{n} = P\hat{z} \cdot \hat{z} = P$

btm: $\sigma_b = \vec{P} \cdot \hat{n} = P\hat{z} \cdot (-\hat{z}) = -P$

Know that for a sheet of charge $E = \frac{\sigma}{2\varepsilon_0}$

$$\therefore \quad \vec{E} = \begin{cases} -\frac{\vec{P}}{\varepsilon_0}, & \text{inside.} \\ 0, & \text{outside} \end{cases}$$

When a neutral object is polarized, the charge moves around a bit, but the object remains neutral.

$$Q_b = \oint \sigma_b \, da + \int \rho_b \, d\tau$$

$$= \oint \vec{P} \cdot \underbrace{\hat{n} \, da}_{d\vec{a}} - \int \vec{\nabla} \cdot \vec{P} \, d\tau$$

$$= \oint \vec{P} \cdot d\vec{a} - \oint \vec{P} \cdot d\vec{a} = 0 \checkmark$$

4.3 The Electric Displacement

We can now find the total electric field. That that cause the polarization in first place in addition to the $\vec{E}$ due to $\vec{P}$.

The object may be charge. In this case, the total charge density ρ is due to (1) the bound charge ρ_b and (2) the "free charge" ρ_f.

any charge not associated w/ $\vec{P}$.

$$\rho = \rho_b + \rho_f$$

Start w/ Gauss's Law:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\therefore \quad \varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho_b + \rho_f$$

$$= -\vec{\nabla}\cdot\vec{P} + \rho_f$$

$$\therefore \quad \vec{\nabla}\cdot\underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{\equiv \vec{D}} = \rho_f$$

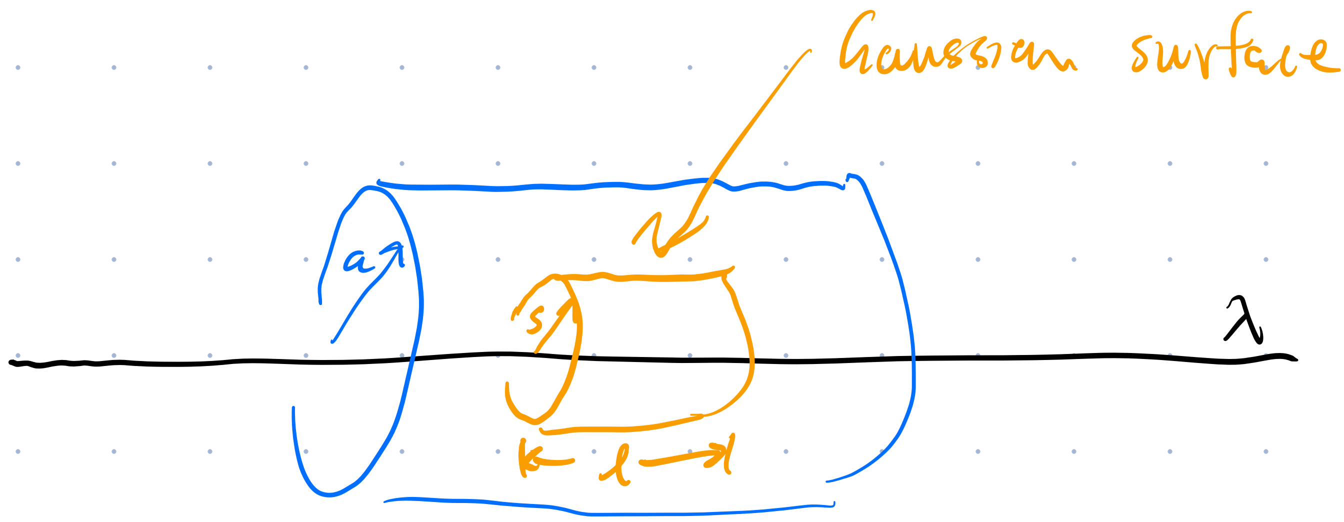
$\equiv \vec{D}$ the electric displacement.

$$\boxed{\therefore \vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow \oint \vec{D}\cdot d\vec{a} = Q_{f,\text{encl}}}$$

Gauss's Law for polarized matter.

Eg. A thin wire w/ charge per unit length λ that is surrounded by rubber insulation of radius a.

Find $\vec{D}$.



$$\oint \vec{D}\cdot d\vec{a} = Q_{f,\text{enc}}$$

$$D\,2\pi s\cancel{l} = \lambda\cancel{l}$$

$$\therefore \vec{D} = \frac{\lambda}{2\pi s}\hat{s}$$ valid inside & outside rubber.

$$\varepsilon_0\vec{E} + \vec{P} = \frac{\lambda}{2\pi s}\hat{s}$$

To find $\vec{E}$ inside the rubber, we would need to know $\vec{P}$. However, outside the rubber, know $\vec{P}=0$

$$\therefore E_{\text{outside}} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s} \qquad s > a$$

Note that $\vec{D}$ is <u>not</u> an exact parallel of $\vec{E}$.

For example, consider:

$$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \left(\varepsilon_0 \vec{E} + \vec{P} \right)$$
$$= \varepsilon_0 \underbrace{\vec{\nabla} \times \vec{E}}_{\substack{0 \text{ in} \\ \text{electrostatics}}} + \vec{\nabla} \times \vec{P}$$

In general, $\vec{\nabla} \times \vec{P} \neq 0 \quad \therefore \vec{\nabla} \times \vec{D} \neq 0$.

→ no scalar potential for $\vec{D}$.

The boundry conditions for $\vec{E}$ were:

$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$$

In terms of $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$:

$$D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

Eg. A thick spherical dielectric shell has a "frozen-in" polarization

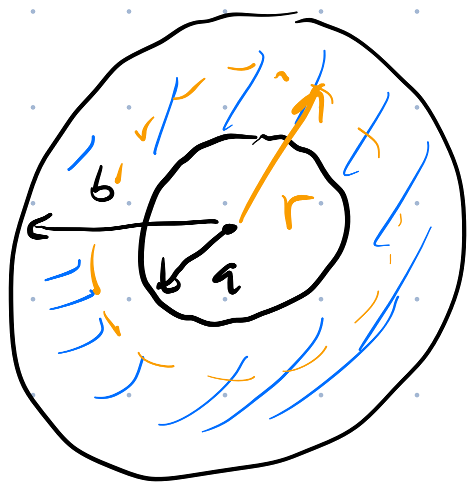
$$\vec{P} = \frac{k}{r}\hat{r}$$

Find $\vec{E}$ in the regions:

$r < a$

$a < r < b$

$r > b$



$$\oint \vec{D} \cdot d\vec{a} = Q_{f,\,encl.}$$

If dielectric is uncharged, $Q_f = 0$.

$$\oint \vec{D} \cdot d\vec{a} = D\,4\pi r^2$$ true $\forall$ 3 regions.

$$= (\varepsilon_0 E + P)\,4\pi r^2 = 0$$

$r < a$: inside cavity of dielectric where $P = 0$

$$\therefore E = 0$$

$r > b$: outside the dielectric where $P = 0$

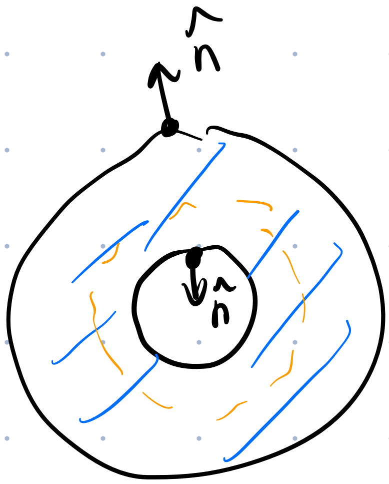
$$\therefore E = 0.$$

$a < r < b$: $(\varepsilon_0 E + P)\,4\pi r^2 = 0$

$$\therefore \varepsilon_0 E = -P \quad E = -\frac{P}{\varepsilon_0}$$

$$\boxed{\therefore \vec{E} = -\frac{k}{\varepsilon_0 r}\hat{r}}$$ $a < r < b$.

Could repeat this calc. using bound charges.



$$\sigma_{b,\text{outside}} = \vec{P}\cdot\hat{n} = \frac{k}{r}\hat{r}\cdot\hat{r} = \frac{k}{r}$$

$$\sigma_{b,\text{inside}} = \hat{P}\cdot\hat{n} = \frac{k}{r}\hat{r}\cdot(-\hat{r}) = -\frac{k}{r}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = \frac{-1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2} \quad a \leq r \leq b$$