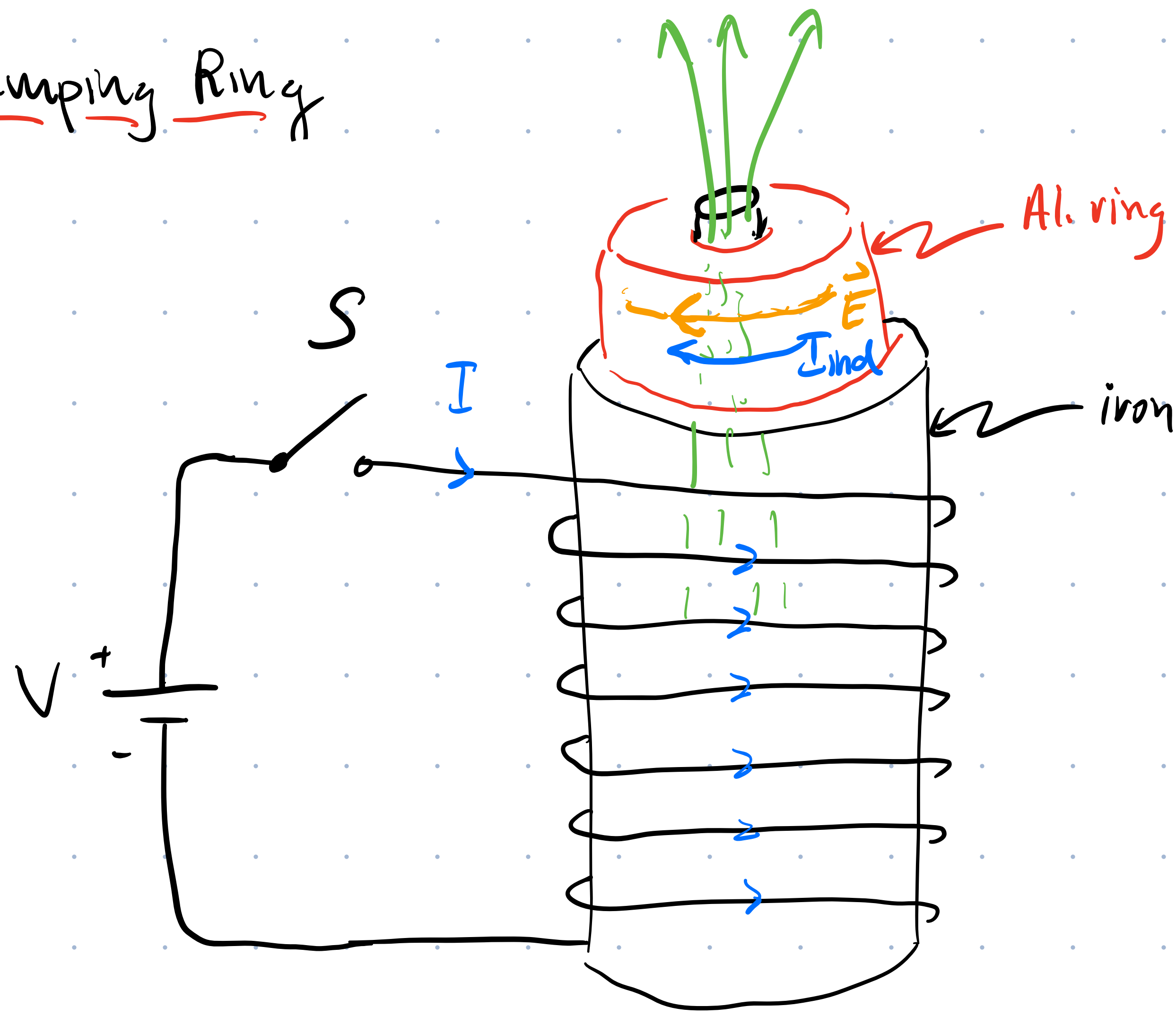
PHYS 301 Nov. 20, 2024

Last Time:

Induced emf: $\mathcal{E} = -\frac{d\Phi}{dt}$

Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Jumping Ring



With switch S, open $I = 0$. Φ through Al. ring is zero. When switch is closed, get a large current through solenoid & a sudden increase of flux through Al. ring.

$$\frac{d\Phi}{dt} > 0$$

This change in flux induces an emf & current in Al. ring.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

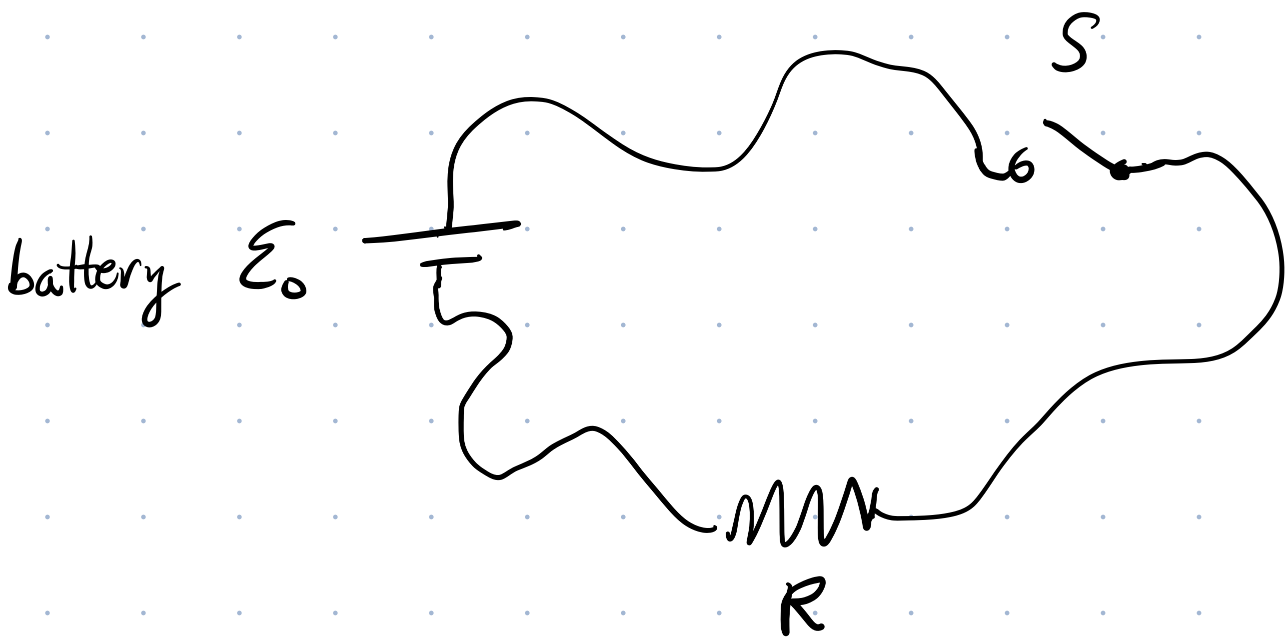
$\frac{\partial \vec{B}}{\partial t}$ is in $+\hat{z}$-dir'n

$\therefore$ $\vec{\nabla} \times \vec{E}$ is in $-\hat{z}$-dir'n.

Induced current in Al. ring that is antiparallel to solenoid current.

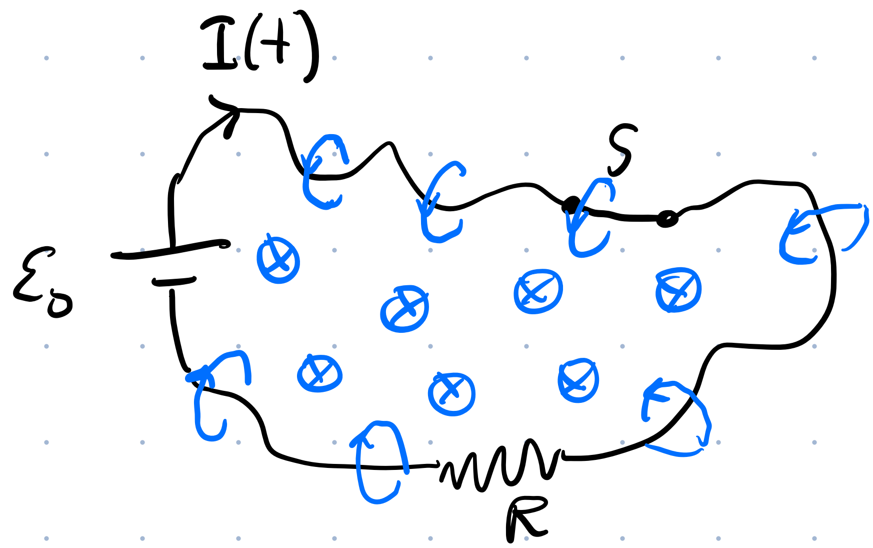
$\therefore$ currents repel one another & ring jumps.

Consider the following "simple" circuit.



Initially, with switch open for a long time, $I = 0$.
Close switch at $t = 0$. Eventually, the current reaches its equil. value $I = \frac{\mathcal{E}_0}{R}$

How does I go from 0 to $\frac{\mathcal{E}_0}{R}$ & in what length of time?



Get a changing magnetic flux $\frac{d\Phi}{dt}$ through circuit loop.

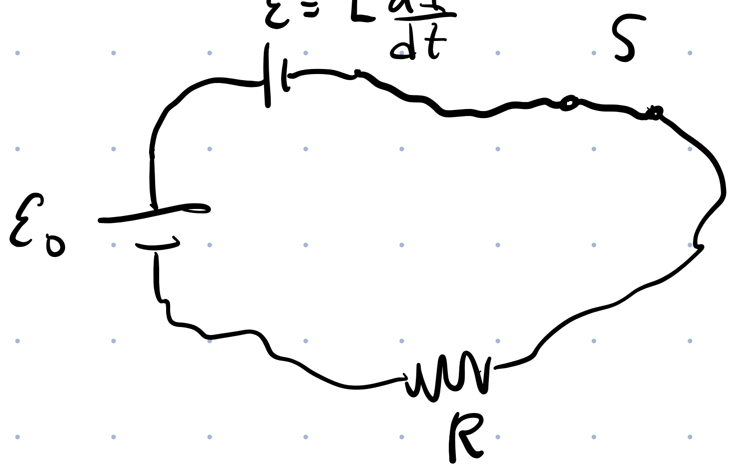
Would be difficult to calc. Φ for a loop of arbitrary geometry. However, it is clear that $\vec{B}$ & Φ are prop. to I.

$$\boxed{\Phi(t) = L\, I(t)}$$

the constant of prop. is called the inductance L.

$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

The equiv. circuit becomes



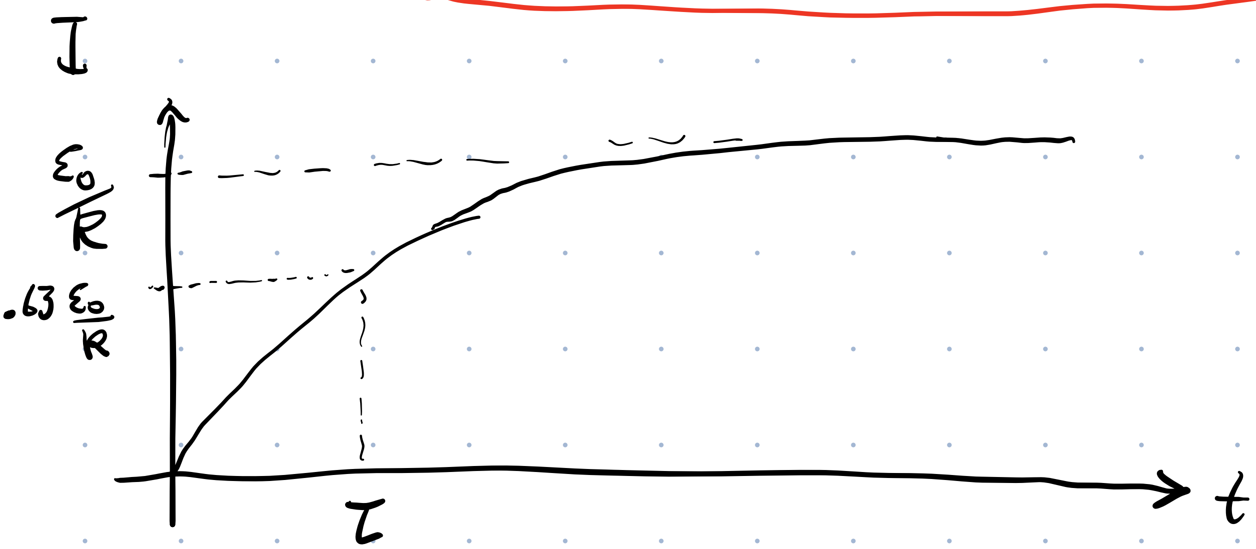
$\mathcal{E}$ is called the "back emf". Non-zero when I is changing.

$$\mathcal{E}_0 - \mathcal{E} - IR = 0$$

or $$L\frac{dI}{dt} + IR = \mathcal{E}_0$$

$$\underbrace{\frac{L}{R}}_{\tau}\frac{dI}{dt} + I = \frac{\mathcal{E}_0}{R} \Rightarrow \tau\frac{dI}{dt} + I = \frac{\mathcal{E}_0}{R}$$

soln is: $$\boxed{I = \frac{\mathcal{E}_0}{R}\left(1 - e^{-t/\tau}\right) \qquad \tau = \frac{L}{R}}$$



How much work does it require to move a charge around the circuit against the back emf?

$$P = \frac{dW}{dt} = I\mathcal{E} = IL\frac{dI}{dt}$$

but $$I\frac{dI}{dt} = \frac{1}{2}\frac{d}{dt}(I^2)$$

$$\therefore \frac{dW}{\cancel{dt}} = \frac{1}{2}L\frac{d}{\cancel{dt}}(I^2)$$

$$\therefore \boxed{W = \frac{1}{2}LI^2}$$

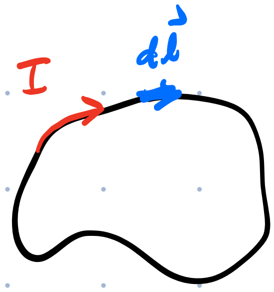
Energy in Magnetic Fields

Recall that $\Phi = LI = \int \vec{B}\cdot d\vec{a}$

$$= \int \vec{\nabla}\times\vec{A}\cdot d\vec{a}$$

$$= \oint_{\text{loop}} \vec{A}\cdot d\vec{l}$$

$$W = \frac{1}{2} I \oint_{\text{loop}} \vec{A} \cdot d\vec{l}$$



$$= \frac{1}{2} \oint (\vec{A} \cdot \vec{I})\, dl$$

This is the result for a 1-D current in a loop.

For volume currents, the analogous expression is

$$W = \frac{1}{2} \int (\vec{A} \cdot \vec{J})\, d\tau$$

From Ampère's Law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$$W = \frac{1}{2\mu_0} \int \vec{A} \cdot (\vec{\nabla} \times \vec{B})\, d\tau$$

By product rule (6)

$$\vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \underbrace{\vec{B} \cdot (\underbrace{\vec{\nabla} \times \vec{A}}_{\vec{B}})}_{B^2} - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

$$\therefore \quad W = \frac{1}{2\mu_0}\left[\int B^2 \, d\tau - \underbrace{\int \vec{\nabla}\cdot(\vec{A}\times\vec{B})\, d\tau}_{\oint_S (\vec{A}\times\vec{B})\cdot d\vec{a}}\right]$$

$$W = \frac{1}{2\mu_0}\left[\int_V B^2 \, d\tau - \oint_S (\vec{A}\times\vec{B})\cdot d\vec{a}\right]$$

If we integrate over all space, then the surface bounding the volume is @ ∞ where both $\vec{A}$ & $\vec{B} \to 0$.

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 \, d\tau \qquad \text{c.t.} \qquad W = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} E^2 \, d\tau}$$

Maxwell's Eq'ns as of now.

(i) $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$ Gauss's Law

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ no name

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's Law

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ Ampère's Law

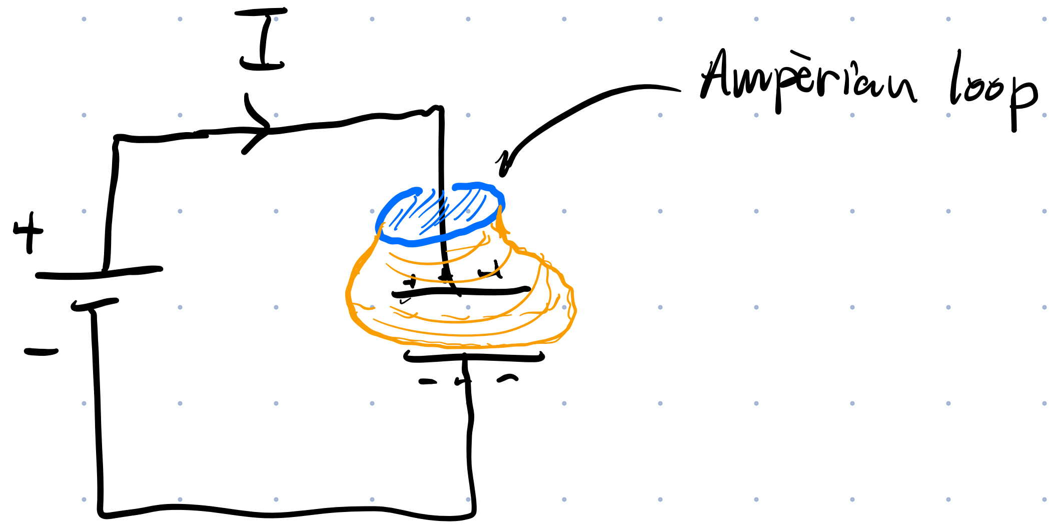
This set of eq'ns is flawed. We can see flaw by take the divergence of (iv).

$$\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}_{0} = \mu_0 \vec{\nabla} \cdot \vec{J}$$

zero only for steady currents.

Fixing (iv).

Imagine charging a capacitor (I is not steady)



Apply Ampère's Law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

For the blue surface bounded by our loop, $I_{encl} = I$.

For the orange surface bounded by the loop $I_{encl} = 0$.!

It's no surprise that Ampère's failed since it applies only to steady currents.

The fix to (iv) is acheived by forcing the RHS to be zero.

Recall continuity eq'n

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

(If $\vec{\nabla} \cdot \vec{J} > 0$, then our volume is losing charge)

Using Gauss's Law we can write

(i) $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0 \quad \therefore \quad \rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$

Then
$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial}{\partial t}\left[\varepsilon_0 \vec{\nabla} \cdot \vec{E}\right]$$
$$= -\vec{\nabla} \cdot \underbrace{\left[\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right]}_{\vec{J}_d}$$

$\vec{J}_d$: displacement current.

The fix to (iv) for the case of time-dependent currents is to add $\vec{J}_d$ to $\vec{J}$.

we had : $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

now : $\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \vec{J}_d \right) = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

Observations

ⓐ In the static case ($\vec{E}$ = const), we recover Ampère's Law for magnetostatics

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

ⓑ $$\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}_{0} = \mu_0 \underbrace{\vec{\nabla} \cdot \vec{J}}_{-\frac{\partial \rho}{\partial t}} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \Big(\underbrace{\vec{\nabla} \cdot \vec{E}}_{\rho/\varepsilon_0} \Big)$$

$$= \mu_0 \frac{\partial \rho}{\partial t} - \mu_0 \frac{\partial \rho}{\partial t} = 0 \checkmark$$

ⓒ The displacement current nicely symmetrizes Maxwell's eq'ns.

Know that $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ from Faraday

Now: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Changing $\vec{B} / \vec{E}$ induces an $\vec{E} / \vec{B}$.

Maxwell's Eq'ns:

(i) $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$ Gauss's Law

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ no name

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's Law

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ corrected Ampère's La

Maxwell's Eq'ns plus the Lorentz force law

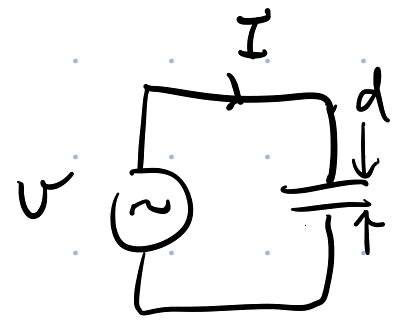
$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}$$

encapsulates all of electrodynamics (outside of matter)

In case you find the displacement current unfamiliar, let's try to show that we've already been using it for some time.

Return to cap. example, only this time, let's drive the cap w/ an AC voltage.
That means the $\vec{E}$ between the cap. plates is osc. & has the form:

$$\vec{E} = \vec{E}_0 e^{j\omega t}$$



cap. plates have area A.

Since there can be no charge transferred across cap. gap., it must be the case that $I = I_d$ between cap plates.

$$J_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{E}_0 e^{j\omega t}\right)$$

$$\therefore \vec{J}_d = j\omega \varepsilon_0 \vec{E}$$

$$\vec{J}_d = \frac{\vec{I}_d}{A} = j\omega \varepsilon_0 \frac{\vec{E}\, d}{d}$$

but $Ed = V_c$: volt. across the cap.

$$\frac{I_d}{A} = j\omega \frac{\varepsilon_0}{d} V_c$$

$$\therefore I_d = j\omega \underbrace{\varepsilon_0 \frac{A}{d}}_{C} V_c$$

$$\therefore I_d = j\omega C V_c$$

or $V_c = \dfrac{I_d}{j\omega C} = I_d Z_c$ ✓

capacitor impedance.