PHYS 301 Nov. 6, 2024

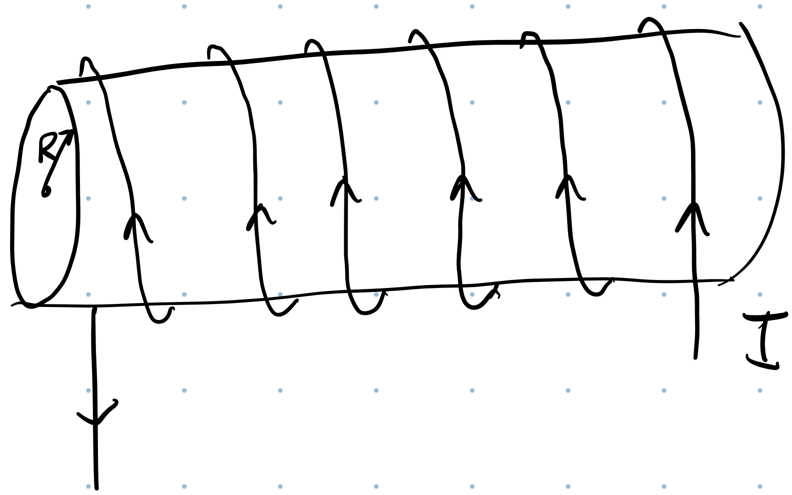
Last Time

Ampère's Law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \iff \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \iff \oint \vec{B} \cdot d\vec{a} = 0$$

Solenoid

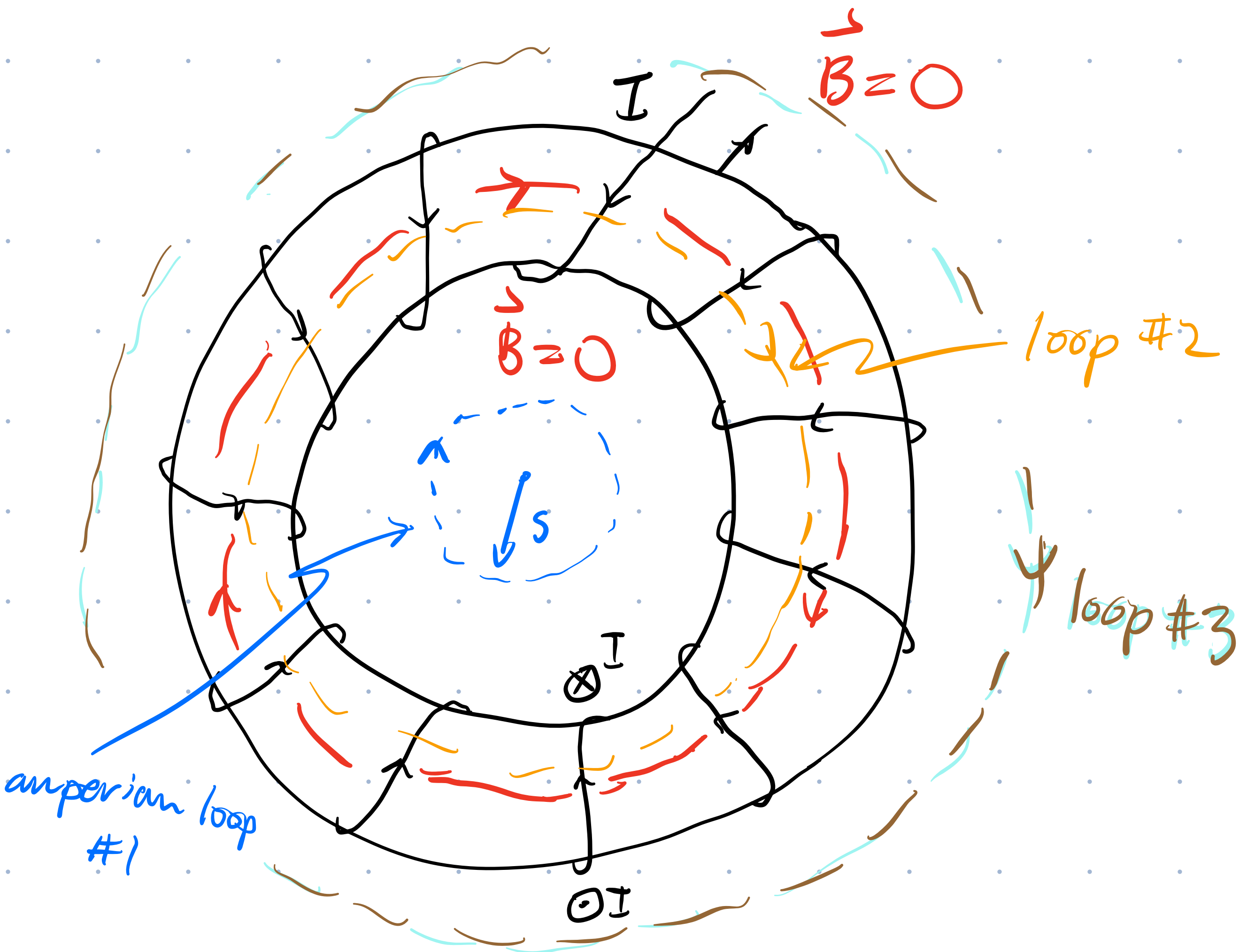


$s < R$ (inside solenoid) $\quad B = \mu_0 n I$

$s > R$ (outside solenoid) $\quad B = 0$

Today: Another Ampère's Law Example:

The toroid



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

$$\vec{B} \parallel d\vec{l} \Rightarrow \vec{B}\cdot d\vec{l} = B\,dl$$

Expect B to const. everywhere on a loop.

$$\therefore \oint \vec{B} \cdot d\vec{l} = \oint B\,dl = B \oint dl = B\,2\pi s$$

true $\forall$ 3 loops.

- Find I_{encl}:

Loop #1: $I_{encl} = 0$

$$\therefore B\,2\pi s = \mu_0(0) \Rightarrow B = 0$$

Loop #3: $I_{encl} = 0$ (equal current into & out of screen)

$$\therefore B = 0$$

Loop #2 $I_{encl} = NI$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} \Rightarrow B\,2\pi s = \mu_0 NI$$

magnetic field of toroid.

$$\boxed{\vec{B} = \frac{\mu_0 NI}{2\pi s}\hat{\phi}}$$

Note that, unlike the solenoid, $\vec{B}$ is <u>not</u> uniform in the toroid. $B \propto \frac{1}{s} \Rightarrow \vec{B}$ is stronger at the inner radius of toroid.

5.4 Magnetic Vector Potential

We know $\vec{\nabla} \cdot \vec{B} = 0$ & $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \ \forall \vec{A}$, it seems like it might be possible to express $\vec{B} = \vec{\nabla} \times \vec{A}$.

We will call $\vec{A}$ the vector potential.

Let's check $\vec{\nabla} \times \vec{B} = \underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{A})}_{\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}} = \mu_0 \vec{J}$

We have some freedom in how we select $\vec{A}$.
It would be nice if we can choose $\vec{A}$ s.t. $\vec{\nabla}\cdot\vec{A}=0$

Aside: Recall $\vec{E} = -\vec{\nabla} V$

$$V \to V_0 + \overset{\text{const.}}{c}$$

$$\vec{\nabla}(V_0 + c) = \vec{\nabla} V_0$$

Add c to the potential leaves $\vec{E}$ unchanged

Since $\vec{\nabla} \times \vec{\nabla}\lambda = 0 \quad \forall \; \lambda$

we can write $\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda$

In this way, $\vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{A}_0 + \vec{\nabla}\lambda)$

$$= \vec{\nabla}\times\vec{A}_0 + \underbrace{\cancel{\vec{\nabla}\times\vec{\nabla}\lambda}}_{0} = \vec{B}$$

Adding $\vec{\nabla}\lambda$ to $\vec{A}_0$ does not change $\vec{B}$.

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\left(\vec{A}_0 + \vec{\nabla}\lambda\right) = \vec{\nabla}\cdot A_0 + \nabla^2\lambda = 0$$

$$\therefore \boxed{\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0}$$ Called the Coulomb gauge.

Notice that $\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0$ is similar to Poisson's eq'n:

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{\mathscr{r}}\,d\tau'$$ Valid for $V \to 0$ @ ∞.

In the same way, if $\lambda \to 0$ @ ∞, it must be the case that

$$\lambda = \frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}_0}{\mathscr{r}}\,d\tau'$$

Weather or not $\lambda \to 0$ @ ∞, it is always possible to find $\vec{A}$ s.t. $\vec{\nabla}\cdot\vec{A}=0$, in which case:

$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$$

3 copies of Poisson's eq'n (one for each component of the vectors),

So, again, if $\vec{J}=0$ @ ∞

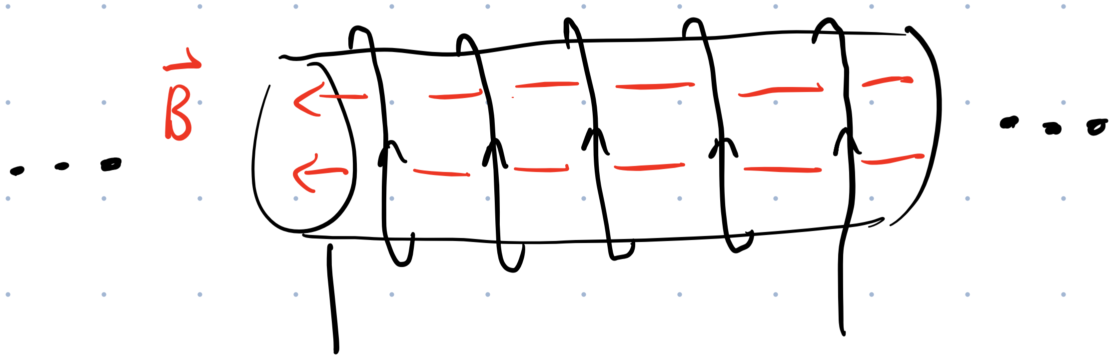
Note that $\vec{A}$ is typically in dir'n of current.

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'}$$

valid if $\vec{J}=0$ @ ∞.

If $\vec{J}\neq 0$ @ ∞, then need to use alternative methods to find $\vec{A}$. Let's see an example.

Eg. Find vector potential of an infinite solenoid.



For infinitely long solenoid, $I \neq 0$ @ $\pm\infty$.

Can't use $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{\mathscr{r}} d\tau'$.

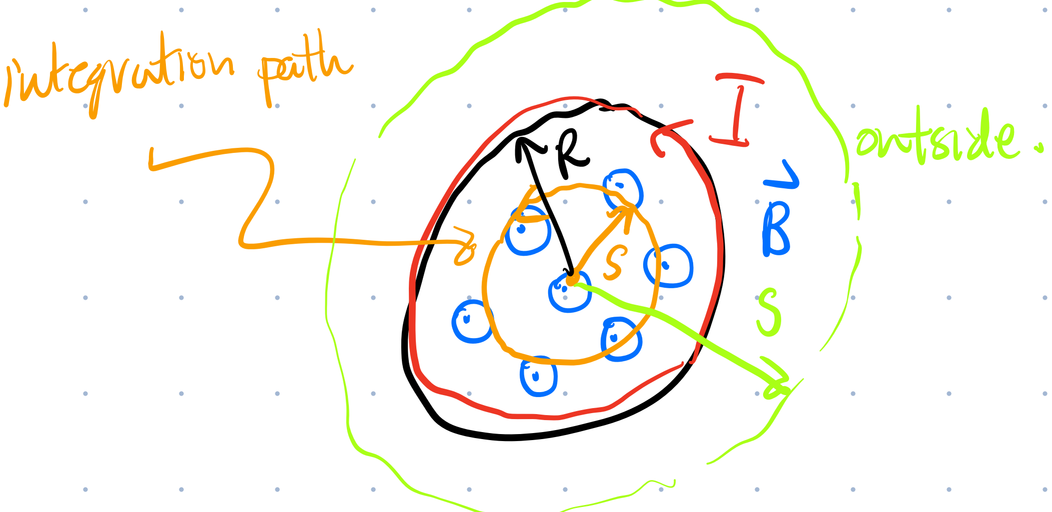
Consider $\oint \vec{A} \cdot d\vec{l} = \underbrace{\int \vec{\nabla} \times \vec{A} \cdot d\vec{a}}_{\text{stoke's Th.}}$

$= \int \vec{B} \cdot d\vec{a} = \Phi$ magnetic flux.

$\Rightarrow \oint \vec{A} \cdot d\vec{l} = \Phi$ Similar to Ampère's Law.

→ Pick an integration path & eval. $\oint \vec{A} \cdot d\vec{l}$ & flux through path.

End View of Solenoid



$$\oint \vec{A}\cdot d\vec{l} = \oint A\,dl = A\oint dl = A\,2\pi s$$

Flux through loop/integration path

$$\Phi = B\pi s^2 = \mu_0 n I \pi s^2$$

$$\oint \vec{A}\cdot d\vec{l} = \Phi \Rightarrow A\,2\cancel{\pi}\cancel{s} = \mu_0 n I \cancel{\pi} s^{\cancel{2}}$$

$$\boxed{\vec{A} = \frac{\mu_0 n I s}{2}\,\hat{\phi}}$$

$s < R$

inside solenoid.

check $\vec{\nabla}\times\vec{A} = \vec{B}$

$$\vec{\nabla}\times\vec{A} = \underbrace{-\cancel{\frac{\partial A_\phi}{\partial z}}}_{0}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}\left(sA_\phi\right)\hat{z}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left[\frac{\mu_0 n I s^2}{2}\right]\hat{z}$$

$$= \cancel{\frac{1}{s}} \mu_0 n I \cancel{s} \, \hat{z} = \mu_0 n I \, \hat{z}$$

$$= \vec{B} \quad ✓$$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_\phi}{\partial \phi} = 0 \quad ✓$$

Outside $s > R$, still have

$$\oint \vec{A} \cdot d\vec{l} = A \, 2\pi s$$

$$\Phi = \mu_0 n I \pi R^2$$

$$A \, 2\cancel{\pi} s = \mu_0 n I \cancel{\pi} R^2$$

$$\boxed{\vec{A} = \frac{\mu_0 n I R^2}{2s} \hat{\phi}}$$

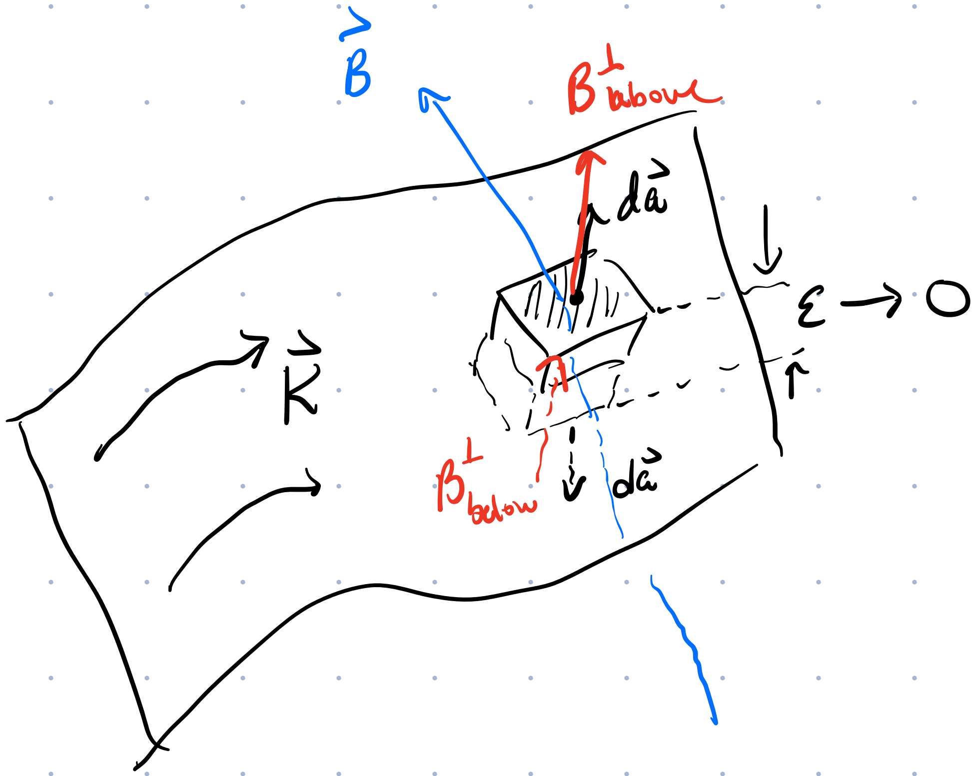
$s > R$
outside solenoid

clearly $\vec{\nabla} \cdot \vec{A} = 0$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\cancelto{0}{\frac{\partial A_\phi}{\partial z}}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}\left(sA_\phi\right)\hat{z}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left[\frac{\mu_0 n I R^2}{2}\right] = 0 \quad \checkmark$$

5.4.2 Boundary Conditions



Consider $\oint \vec{B} \cdot d\vec{a} = 0$

$d\vec{a} \perp$ to surface

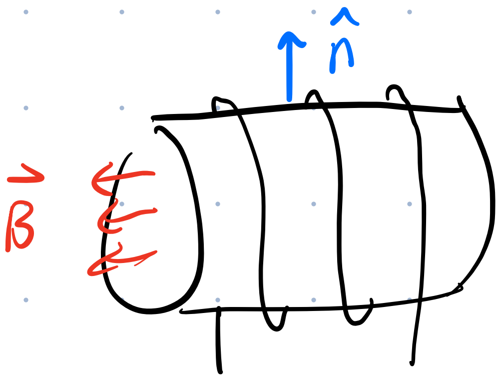
$$\oint \vec{B}\cdot d\vec{a} = \int_{top} \vec{B}\cdot d\vec{a} + \int_{btm} \vec{B}\cdot d\vec{a} = 0$$

$$= \int_{top} B^{\perp}_{above}\, da - \int_{btm} B^{\perp}_{below}\, da = 0$$

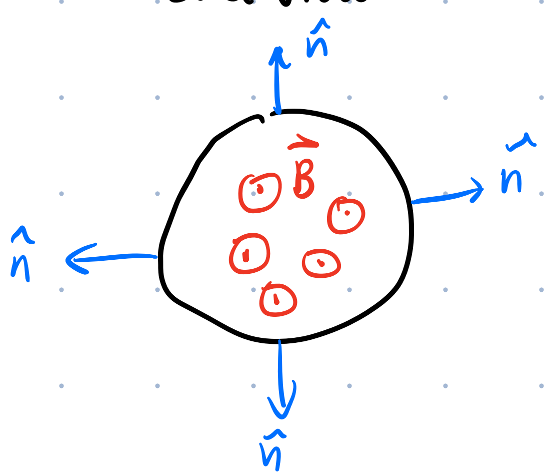
$$= B^{\perp}_{above} \cancel{\int da} - B^{\perp}_{below} \cancel{\int da} = 0$$

$$\boxed{\therefore\ B^{\perp}_{above} = B^{\perp}_{below}}$$

Check for the case of a solenoid.

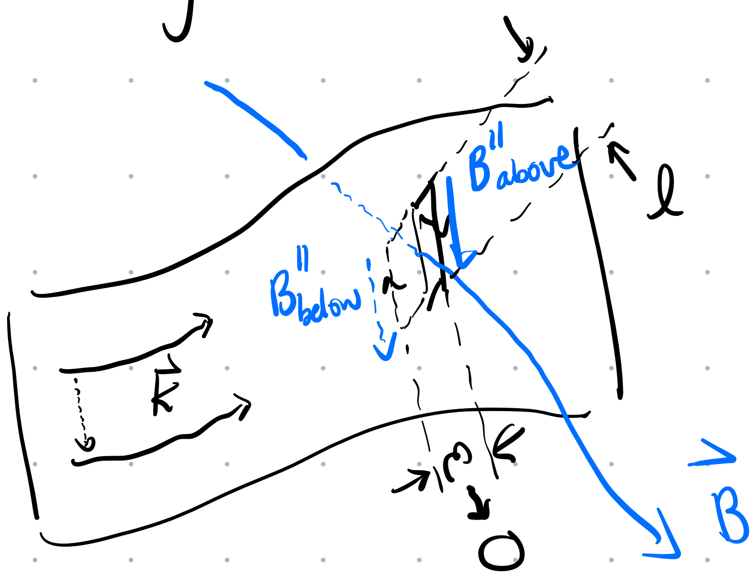


End view



$$B^{\perp}_{in} = B^{\perp}_{out} = 0\ \checkmark$$

Next, consider $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

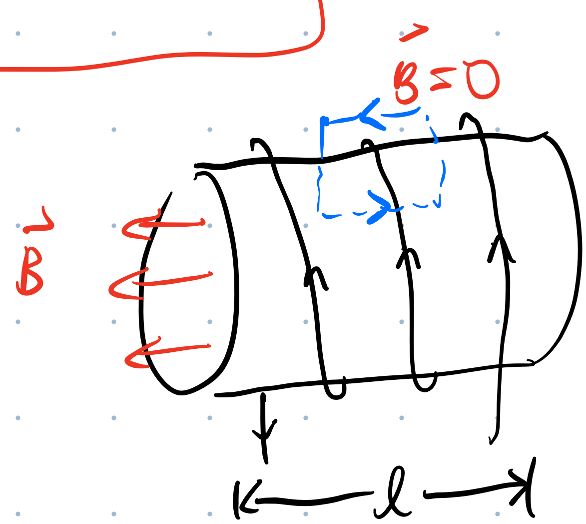


$$\oint \vec{B}\cdot d\vec{l} = \int B^{\parallel}_{above}\, dl - \int B^{\parallel}_{below}\, dl = \mu_0 \underbrace{I_{encl}}_{Kl}$$

$$= B^{\parallel}_{above} \underbrace{\int \cancel{dl}}_{l} - B^{\parallel}_{below} \underbrace{\int \cancel{dl}}_{l} = \mu_0 K \cancel{l}$$

$$\boxed{\therefore \quad B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K}$$

Eg. Solenoid.



$$B^{\parallel}_{out} = 0 \qquad B^{\parallel}_{in} = -\mu_0 \frac{N}{\ell} I$$

$$B^{\parallel}_{out} - B^{\parallel}_{in} = \cancel{\mu_0} \frac{N}{\ell} I = \cancel{\mu_0} K$$

$$K = \frac{NI}{\ell} \quad \checkmark$$

There are two more boundary conditions. They apply to $\vec{A}$.

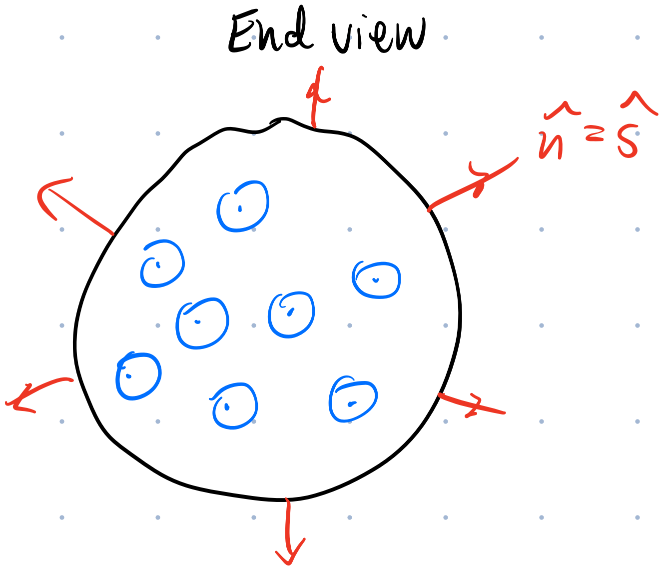
$$\boxed{\begin{gathered} \vec{A}_{above} = \vec{A}_{below} \quad \text{Vector potential is continuous @ surface.} \\[1em] \frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K} \end{gathered}}$$

Check for solenoid.

$$\vec{A}_{in} = \frac{\mu_0 n I s}{2}\hat{\phi} \qquad \vec{A}_{out} = \frac{\mu_0 n I R^2}{2s}\hat{\phi}$$

$$\vec{A}_{in}\Big|_{s=R} = \vec{A}_{out}\Big|_{s=R} \checkmark$$

$$\frac{\partial \vec{A}_{out}}{\partial s}\Big|_{s=R} - \frac{\partial \vec{A}_{in}}{\partial s}\Big|_{s=R} = -\mu_0 \vec{K} \,?$$



$$-\frac{\mu_0 n I R^2}{2s^2}\hat{\phi}\Big|_{s=R} - \frac{\mu_0 n I}{2}\hat{\phi}\Big|_{s=R} = -\mu_0 \vec{K}$$

$$\cancel{-}\frac{\cancel{\mu_0} n I}{2}\hat{\phi} \cancel{-} \frac{\cancel{\mu_0} n I}{2}\hat{\phi} = \cancel{-}\cancel{\mu_0}\vec{K}$$

$$n I \hat{\phi} = \vec{K} \Rightarrow \vec{K} = \frac{N I}{\ell}\hat{\phi} \checkmark$$