PHYS 301 Nov. 4, 2024

Last Time:

Continuity Eq'n: $\vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}$

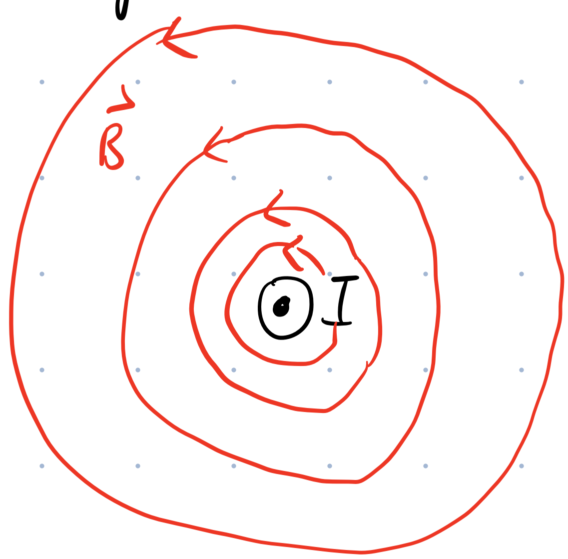
For steady currents, $\frac{d\rho}{dt} = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = 0}$

Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, dl'$$

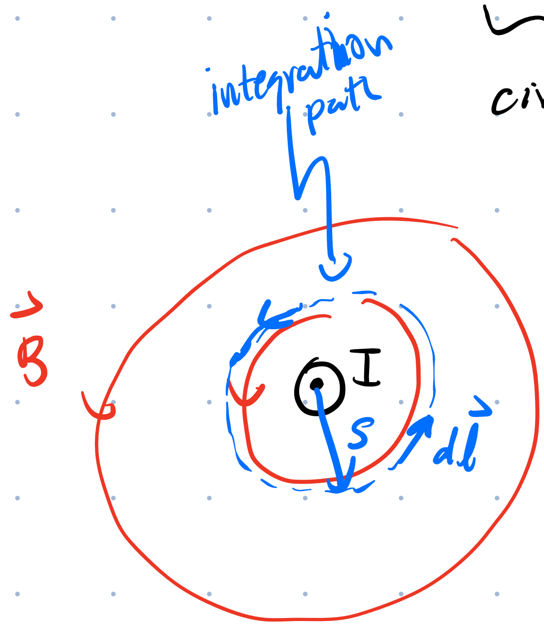
Magnetic field due to a long, straight current:

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$



Today: Griffiths 5.3 The Divergence & Curl of $\vec{B}$

Let's try evaluating $\underbrace{\oint \vec{B} \cdot d\vec{l}}_{\text{circulation}} = \int_S \vec{\nabla} \times \vec{B} \cdot d\vec{a}$ Stoke's Th.



Select an integration path that matches the symmetry of $\vec{B}$.

$\therefore\ d\vec{l} = dl\,\hat{\phi}$ $\quad \vec{B} = \dfrac{\mu_0 I}{2\pi s}\hat{\phi}$

$$\therefore\ \vec{B} \cdot d\vec{l} = \underbrace{\frac{\mu_0 I}{2\pi s}}_{\text{const. on chosen integration path.}} dl$$

$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{\cancel{2\pi s}} \underbrace{\cancel{\oint dl}}_{\text{circumference of integration path}}$$

Integral form of Ampère's Law.

$$\boxed{\therefore \oint \vec{B}\cdot d\vec{\ell} = \mu_0 I}$$

$= \cancel{2\pi} s$

apply stoke's Law

So far, we've shown that this result is true for a long straigh current. We want generalize for $\vec{B}$ due to any steady current dist'n.

Note that, last time we showed $\boxed{\int_S \vec{J}\cdot d\vec{a}} = I$

$$\int_S \underbrace{\vec{\nabla}\times\vec{B}}\cdot d\vec{a} = \int_S \underbrace{(\mu_0 \vec{J})}\cdot d\vec{a}$$

must be equal

$$\boxed{\therefore \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}}$$

differential form of Ampère's Law.

#

The goal is to reproduce (#) starting from the general Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

Integrate over primed coords which locate the source currents w.r.t. the origin. Want to find $\vec{\nabla} \times \vec{B}$ & $\vec{\nabla} \cdot \vec{B}$. The $\vec{\nabla}$ operator takes deriv. w.r.t. unprimed coords. which locate the field pt.

Start w/ $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' \right)$

$$= \frac{\mu_0}{4\pi} \int \underbrace{\vec{\nabla} \cdot \left(\vec{J}(\vec{r}\,') \times \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \right)}_{\text{Apply product rule (6) from Griffiths cover}} d\tau'$$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\left(\cancelto{0}{\vec{\nabla}\times\vec{J}(\vec{r}\,')}\right) - \vec{J}(\vec{r}\,')\cdot\underbrace{\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{0}$$

since der. w.r.t. unprimed coords

since $\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \dfrac{\vec{\mathscr{r}}}{\mathscr{r}^3}$ has no θ or ϕ dependence and no θ or ϕ component in spherical coords.

$$\boxed{\therefore\ \nabla\cdot\vec{B} = 0}$$

True $\forall\ \vec{B}$ due to steady currents.

Next, consider $\vec{\nabla} \times \vec{B}$:

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \underbrace{\vec{\nabla} \times \left(\vec{J}(\vec{r}\,') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)}_{} d\tau'$$

Product rule (8)

$$\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \vec{\nabla} \right) \cancelto{0}{\vec{J}(\vec{r}\,')} - \left(\vec{J}(\vec{r}\,') \cdot \vec{\nabla} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$+ \vec{J}(\vec{r}\,') \underbrace{\left(\vec{\nabla} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)}_{4\pi \delta^3(\vec{\mathscr{r}})} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cancelto{0}{\left(\vec{\nabla} \cdot \vec{J}(\vec{r}\,') \right)}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \underbrace{\int \vec{J}(\vec{r}\,') \, \delta^3(\vec{r} - \vec{r}\,') \, d\tau'}_{\vec{J}(\vec{r})}$$

selects value of $\vec{r}\,'$ s.t. $\vec{r} - \vec{r}\,' = 0$

$\Rightarrow \vec{r} = \vec{r}\,'$

$$\underbrace{- \frac{\mu_0}{4\pi} \int \left(\vec{J}(\vec{r}\,') \cdot \vec{\nabla} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'}_{Int_2}$$

We will show that $\text{Int}_2 = 0$ s.t.

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

Aside: Consider $\dfrac{\partial}{\partial x} f(u)$ where $u = x - x'$

$$= \frac{df}{du} \cancelto{1}{\frac{du}{dx}} = \frac{df}{du}$$

Consider $\dfrac{\partial}{\partial x'} f(u)$

$$= \frac{df}{du} \cancelto{-1}{\frac{du}{dx'}} = -\frac{df}{du}$$

$$\boxed{\frac{\partial}{\partial x'} f(u) = -\frac{\partial}{\partial x} f(u)}$$

By the same logic, $\vec{\nabla} \dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ in Int_2

is equiv. to $-\vec{\nabla}' \dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$\therefore$ Int_2 becomes:

$$-\frac{\mu_0}{4\pi}\int \left(\vec{J}(\vec{r}\,')\cdot\vec{\nabla}'\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

Consider just the x-comp. of $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ is Cartesian coord.

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$\left(\vec{J}(\vec{r}\,')\cdot\vec{\nabla}'\right)\left(\frac{x-x'}{\mathscr{r}^3}\right)$$

Product rule (5)

$$= \vec{\nabla}'\cdot\left(\frac{x-x'}{\mathscr{r}^3}\vec{J}(\vec{r}\,')\right)$$

$$-\left(\frac{x-x'}{\mathscr{r}^3}\right)\underbrace{\vec{\nabla}'\cdot\vec{J}(\vec{r}\,')}$$

0 for steady currents

(continuity eq'n w/ $\frac{d\rho}{dt}=0$)

$\therefore$ the x-component's contribution to Int_z is:

$$\int_V \vec{\nabla}' \cdot \left(\frac{x - x'}{\mathscr{r}^3} \vec{J}(\vec{r}\,') \right) d\tau'$$

Apply the divergence theorem

$$= \oint_S \frac{x-x'}{\mathscr{r}^3} \vec{J}(\vec{r}\,') \cdot d\vec{a}\,'$$

S ← surface bounding the integral in Biot-Savart law that contains $\vec{J}(\vec{r}\,')$.

Choose surface to be large enough that $\vec{J}(\vec{r}\,') = 0$ everywhere on the surface s.t. $Int_z = 0$. (similar for y & z components).

Summary

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} \end{aligned}}$$

Ampère's Law in differential form →

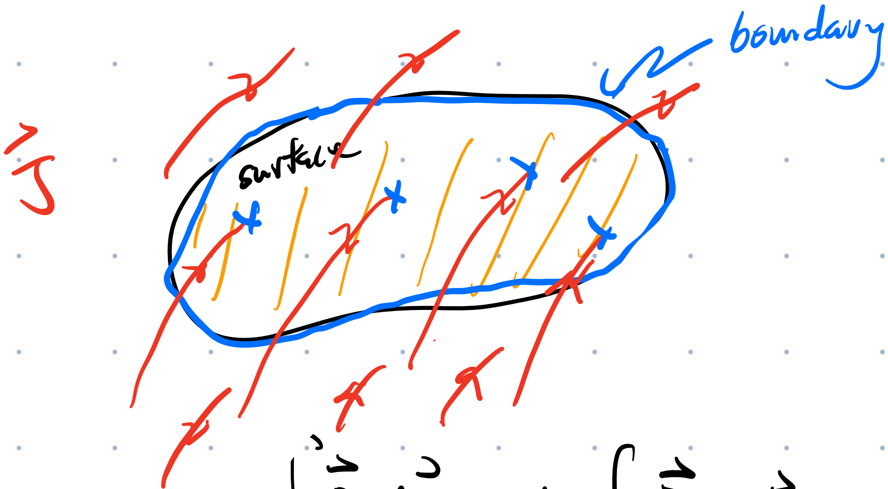
valid ∀ magnetostatics.

5.3.3 Ampère's Law

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

Integrate

$$\int_S \vec{\nabla}\times\vec{B}\cdot d\vec{a} = \oint_{\text{boundary}} \vec{B}\cdot d\vec{l}$$

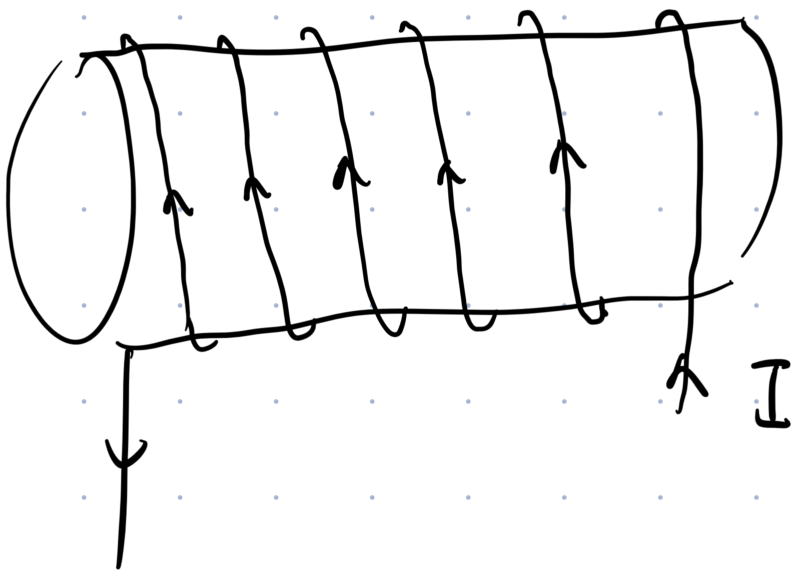


$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{\int \vec{J}\cdot d\vec{a}}_{I_{encl}} = \mu_0 I$$

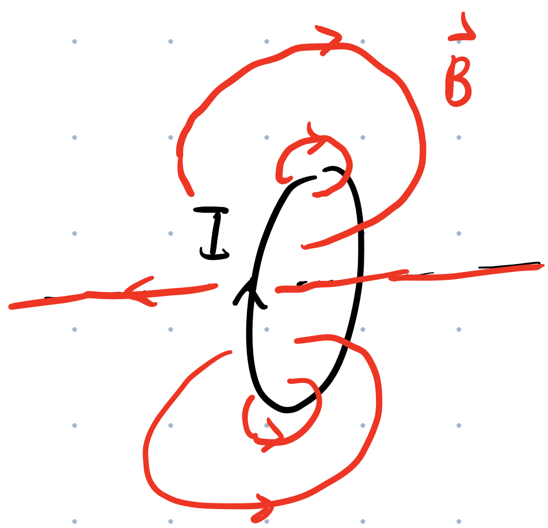
I_{encl} ← total current passing surface S.

$$\boxed{\therefore \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}} \quad \text{integral form of Ampère's Law}$$

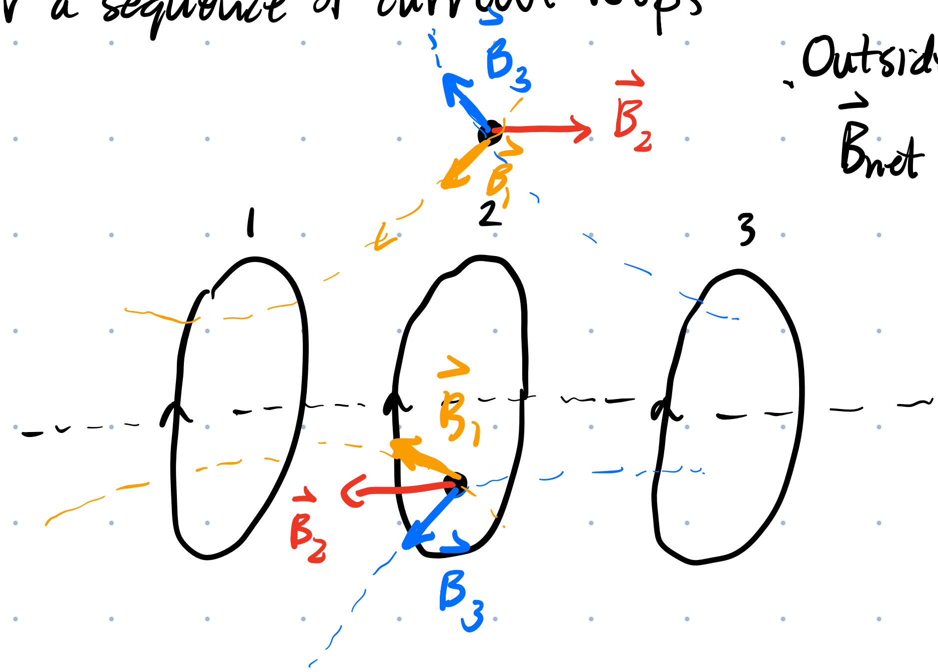
Eg. Find $\vec{B}$ due to a long solenoid.



Like a series of current loops.



Consider a sequence of current loops

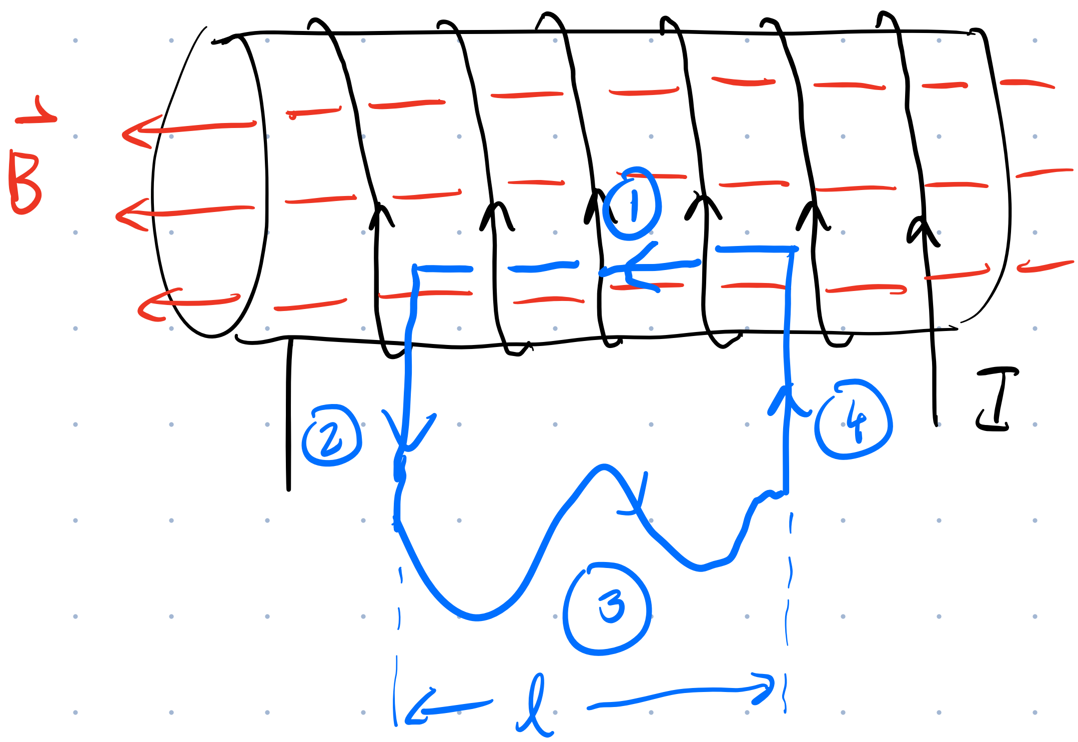


Outside the solenoid $\vec{B}_{net} \approx 0$.

Inside solenoid, horizontal $\vec{B}$-field is strong while the vertical component vanishes

Apply Ampère's to the solenoid.

$$B = 0$$



1. Select an Ampèrian loop/integration path

path: - should be $\parallel$ or $\perp$ to $\vec{B}$

- B should be const. when path $\parallel$ to $\vec{B}$.

2. Find I_{encl} (how much current passes through Ampèrian loop)

N loops of the solenoid coil pass through the loop.

$$\therefore I_{encl} = NI$$

3. Evaluate

$$\oint \vec{B}\cdot d\vec{l} = \int_{①} \vec{B}\cdot d\vec{l} + \underbrace{\int_{②} \vec{B}\cdot d\vec{l}}_{\substack{0 \text{ b/c } \vec{B} \perp d\vec{l} \text{ inside solenoid} \\ \& \ \vec{B}=0 \text{ outside solenoid}}}$$

$$+ \underbrace{\int_{③} \vec{B}\cdot d\vec{l}}_{0 \text{ b/c } \vec{B}=0 \text{ everywhere on ③}} + \underbrace{\int_{④} \vec{B}\cdot d\vec{l}}_{0 \text{ for same reasons as ②}}$$

$$\oint \vec{B}\cdot d\vec{l} = \int_{①} \vec{B}\cdot d\vec{l} = \int B\,dl$$

since $\vec{B} \parallel d\vec{l}$ for ①

$$= B\int dl$$

since ① is same position inside solenoid (same dist. for solenoid axis)

$$= B\,l$$

↑ length of path ①

Finally: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

$$\Downarrow$$

$$Bl = \mu_0 N I$$

$$\therefore B = \mu_0 \left(\frac{N}{l}\right) I$$

n : no. of turns per unit length.

$$\boxed{\therefore B = \mu_0 n I}$$

$\vec{B}$ inside solenoid is <u>uniform</u>.