PHYS 301 Oct. 30, 2024

<u>Last Time:</u>

Moving charges / currents :

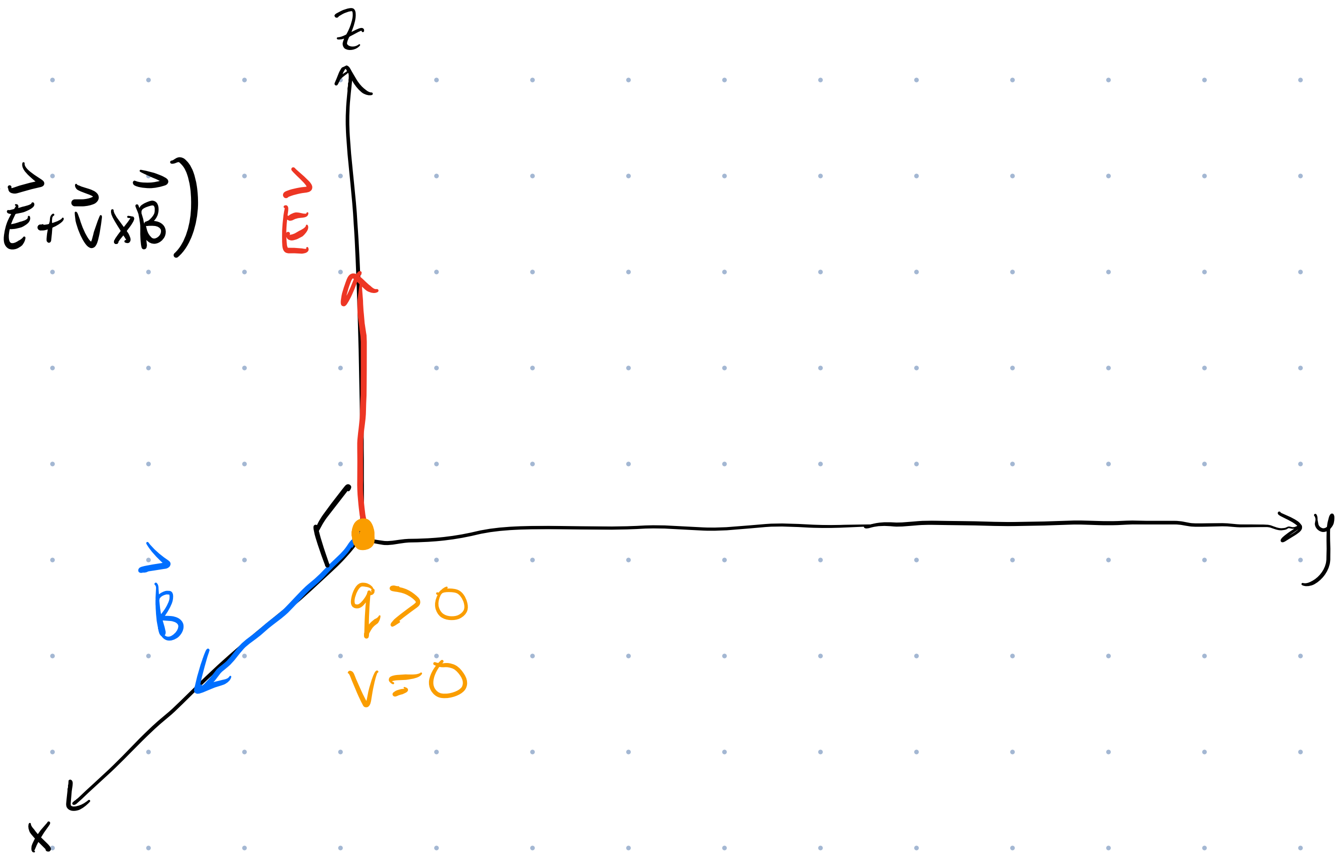
- are a source of magnetic fields.
- experience a force due to other external magnetic fields.

Force on a charge in a region of space w/ both $\vec{E}$ & $\vec{B}$:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

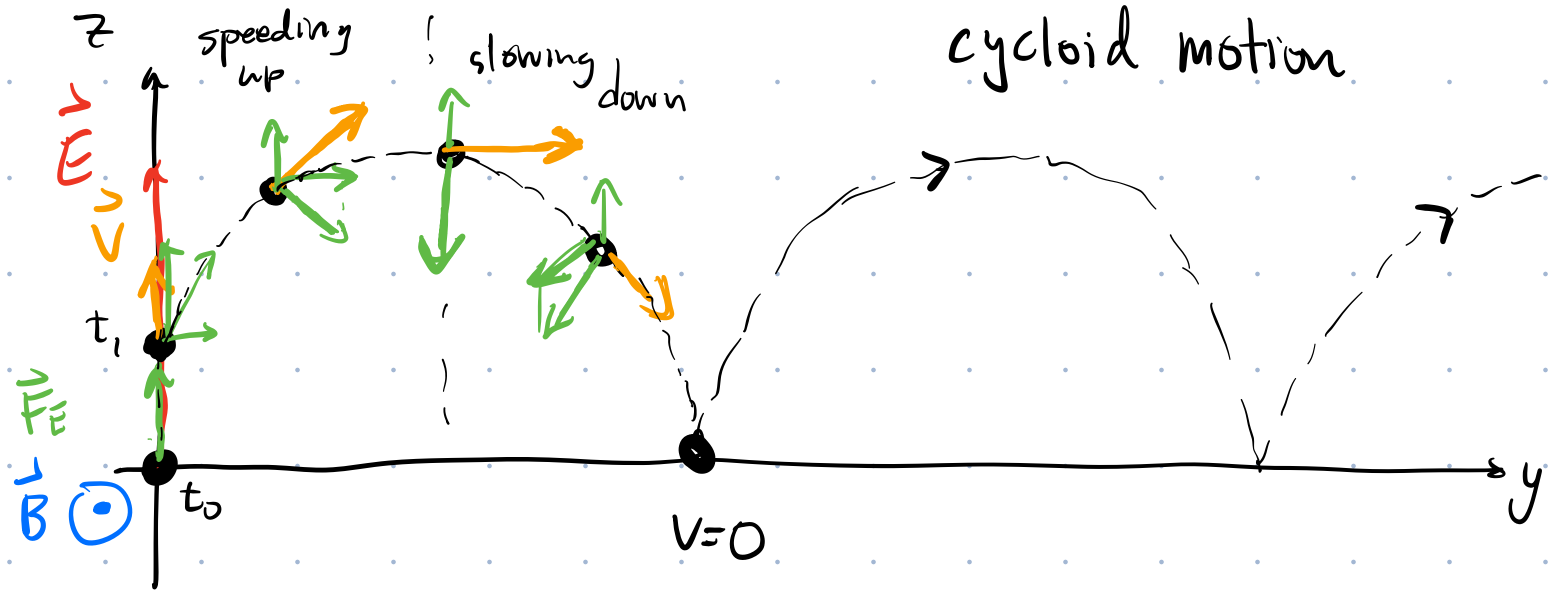
Today, consider the strage example of a particle at rest in a uniform $\vec{E}$ & $\vec{B}$ field, w/ $\vec{E} \perp \vec{B}$.

$$F = q(\vec{E} + \vec{v} \times \vec{B})$$



We will examine the resulting motion of the charge qualitatively. See Griffiths eg. 5.2 for a quantitative analysis.

Start w/ q at rest at origin

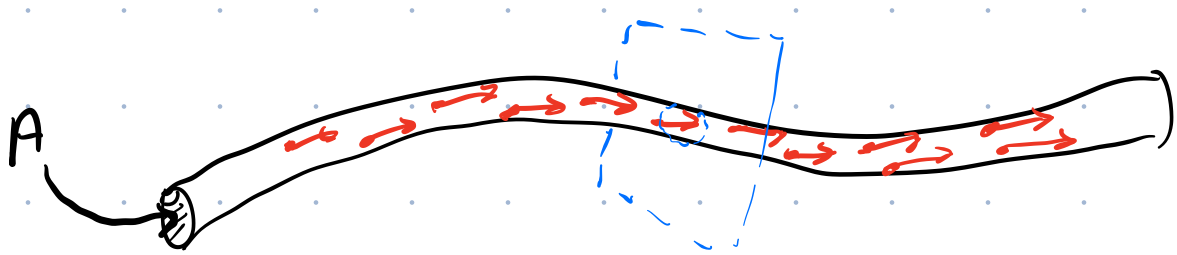


Motion of q is $\perp$ to both $\vec{E}$ & $\vec{B}$.

Get a displacement along axis.

5.1.3 Current

Current is the charge per unit time crossing a surface.



If total charge Δq crosses surface in time Δt

$$I_{avg} = \frac{\Delta q}{\Delta t} \qquad I = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$$

If wire has a free/mobile charge density

charge per unit volume $\rightarrow \rho = e\,n$ ← # electrons per unit volume.

charge of electron

$$\lambda = \frac{Q}{l}\cdot\frac{A}{A} = \frac{QA}{V} = \rho A$$

$$\lambda = \rho A = \frac{\Delta q}{\Delta x}$$

$$\therefore \lambda v = \frac{\Delta q}{\cancel{\Delta x}}\underbrace{\frac{\cancel{\Delta x}}{\Delta t}}_{v} = \frac{\Delta q}{\Delta t} = I$$

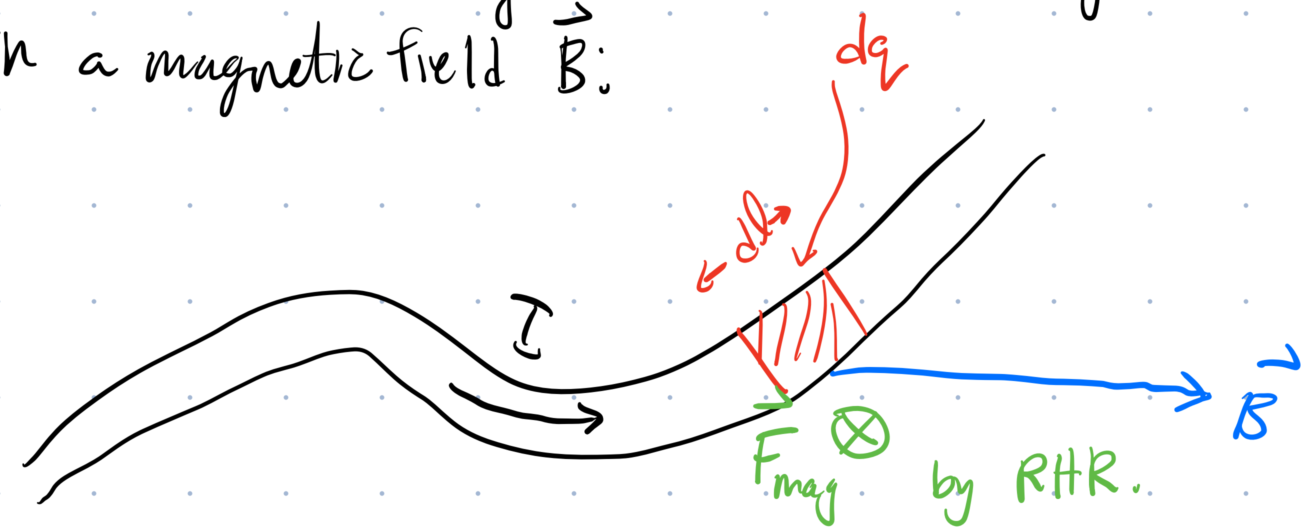
drift velocity

$$\therefore I = \frac{dq}{dt} = \lambda v = \rho A v = enAv$$

If we want, we can define I as a vector

$$\vec{I} = \lambda \vec{v}$$

The force on a segment of wire of length dl in a magnetic field $\vec{B}$:



$$\underset{\text{Force on segment } dl}{d\vec{F}_{mag}} = \underbrace{dq}_{\lambda dl}\, \vec{v} \times \vec{B} = \underbrace{(\lambda \vec{v})}_{\vec{I}} \times \vec{B}\, dl$$

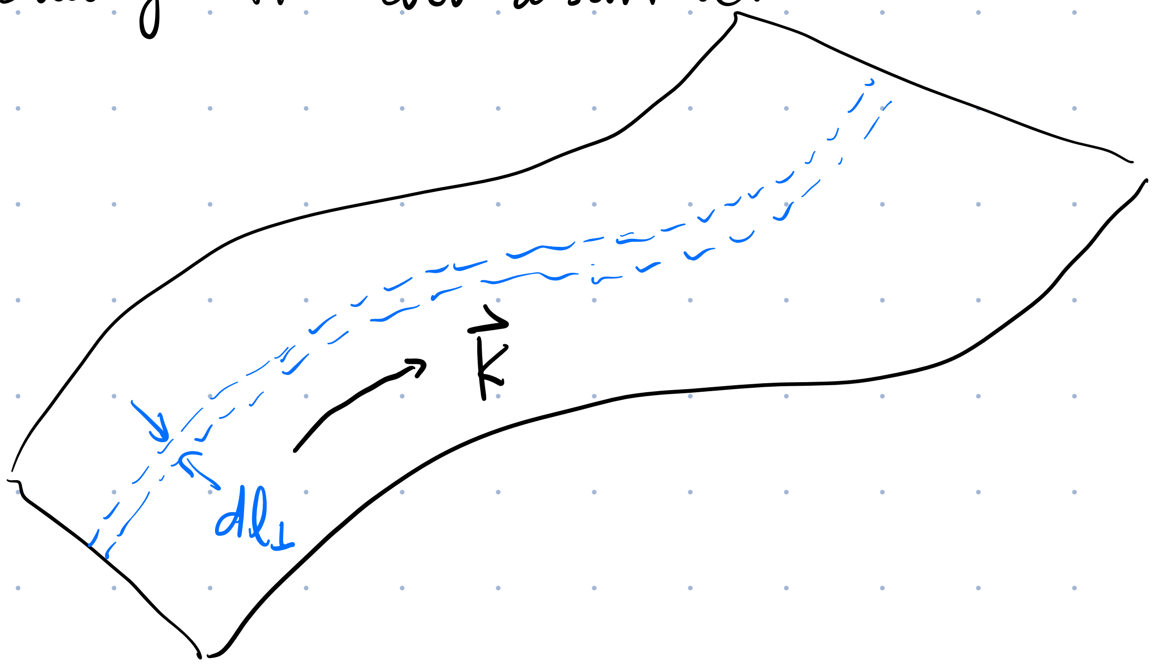
$$\therefore\ d\vec{F}_{mag} = \vec{I} \times \vec{B}\, dl$$

$$\boxed{\vec{F}_{mag} = \int \vec{I} \times \vec{B}\, dl}$$

It is conventional to write $\vec{I}$ as a scalar & assign the dir'n to $dl \to d\vec{l}$

$$\boxed{\vec{F}_{mag} = \int I\, d\vec{l} \times \vec{B}}$$

Can generalize this result to the case of a charge flow over a surface.



$\vec{K}$ is the surface current density

Consider a ribbon of width $dl_\perp$ that is parallel to charge flow. the ribbon carries current

$$d\vec{I} = \vec{K}\, dl_\perp$$

$$\vec{K} = \frac{d\vec{I}}{dl_\perp}$$ current per unit width.

$$d\vec{I} = \lambda \vec{v} = \boxed{\frac{dq}{dx}\vec{v}}$$

charge per area.

$$\vec{K} = \frac{dq}{dx\, dl_\perp}\vec{v} = \frac{dq}{da}\vec{v} = \sigma \vec{v}$$

where $da = dx\, dl_\perp$ wire area.

$$\vec{I} = \lambda \vec{v}$$

$$\vec{K} = \sigma \vec{v}$$

$$dq = \sigma\, da$$

$$dq\, \vec{v} = \sigma \vec{v}\, da$$

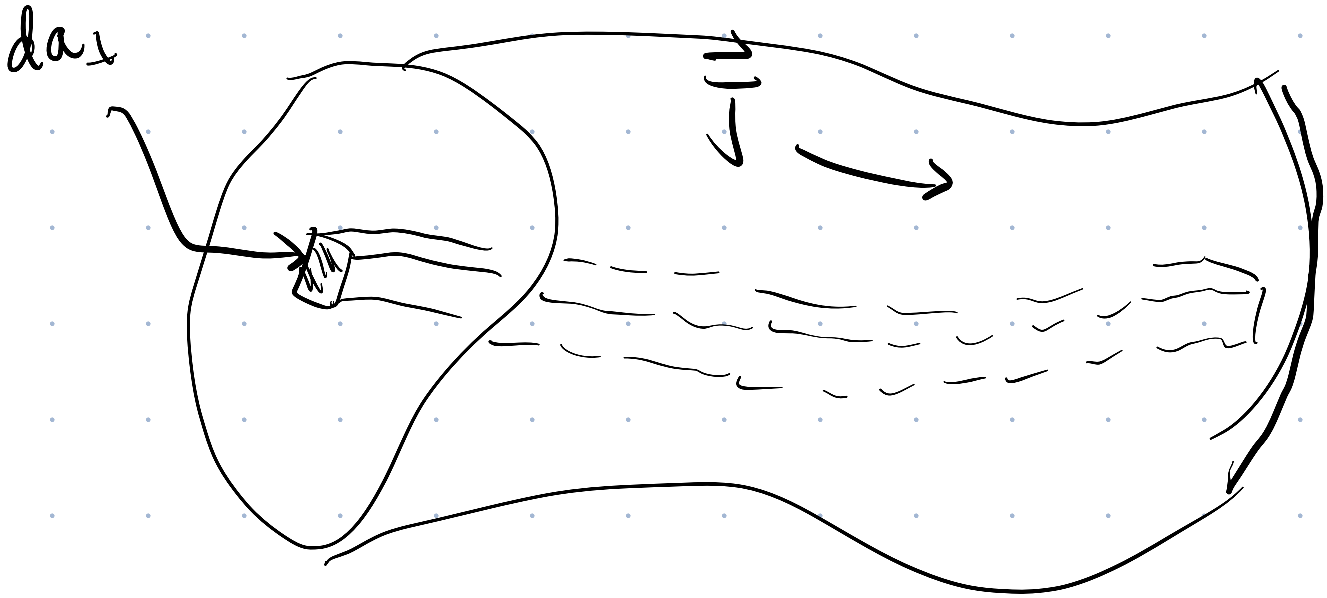
$\therefore$ Force on surface current

$$d\vec{F}_{mag} = dq\, \vec{v} \times \vec{B}$$

$$= \underbrace{(\sigma \vec{v})}_{\vec{K}} \times \vec{B}\, da$$

$$\boxed{F_{mag} = \int \sigma \vec{v} \times \vec{B}\, da = \int \vec{K} \times \vec{B}\, da}$$

Finally, can have a current distributed throughout a volume.



$$d\vec{I} = \vec{J}\, da_\perp \quad \#$$

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \frac{\lambda \vec{v}}{da_\perp} = \frac{dq}{dx}\frac{\vec{v}}{da_\perp} = \overbrace{\frac{dq}{d\tau}}^{\rho}\vec{v}$$

where $d\tau = da_\perp dx$
infinitesimal
volume.

$\frac{dq}{d\tau} = \rho$ charge density.

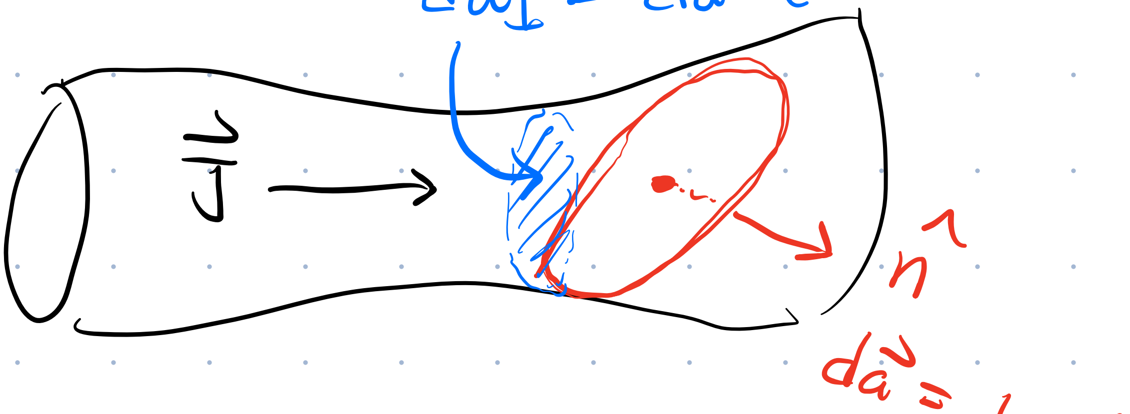
↳ $dq = \rho\, d\tau$ $\quad$ $dq\vec{v} = \rho\vec{v}\, d\tau$

$$d\vec{F}_{mag} = dq\,\vec{v}\times\vec{B} = (\rho\vec{v})\times\vec{B}\,d\tau$$
$$= \vec{J}\times\vec{B}\,d\tau$$

$$\boxed{\therefore\ \vec{F}_{mag} = \int \rho\vec{v}\times\vec{B}\,d\tau = \int \vec{J}\times\vec{B}\,d\tau}$$

Return to ⊕ : $dI = J\,da_\perp$

$da_\perp = d\vec{a}\cdot(\text{dir'n of } \vec{J})$



$$\therefore\ \vec{J}\cdot d\vec{a} = J\,da_\perp$$

Total current cross surface w/ area $d\vec{a}$ is:

$$I = \int J\,da_\perp = \int \vec{J}\cdot d\vec{a}$$

For a closed surface that bounds a volume $\mathcal{V}$ through which $\vec{J}$ flows, the current crossing/exiting that bounding surface is given by:

$$I = \oint_S \vec{J} \cdot d\vec{a}$$

current exiting our volume $\mathcal{V}$.

$$= \int_{\mathcal{V}} \vec{\nabla} \cdot \vec{J} \, d\tau$$

(Divergence Theorem)

If our current is exiting our volume, we're losing charge at a rate $-\frac{dq}{dt}$

$$\int_V (\vec{\nabla}\cdot\vec{J})\,d\tau = -\frac{d}{dt}(q) = -\frac{d}{dt}\underbrace{\int_V \rho\, d\tau}_{q}$$

q: total charge in V.

$$= \int \left(-\frac{d\rho}{dt}\right) d\tau$$

$$\boxed{\therefore \quad \vec{\nabla}\cdot\vec{J} = -\frac{d\rho}{dt}}$$

This is a statement about the conserv. of charge.

It is called the continuity equation.

A steady current is defined as a current which is constant. Any charge exiting a volume is replenished by an equal inward flow of charge.

$$\Rightarrow \frac{d\rho}{dt} = 0 \Rightarrow \text{magnetostatics.}$$

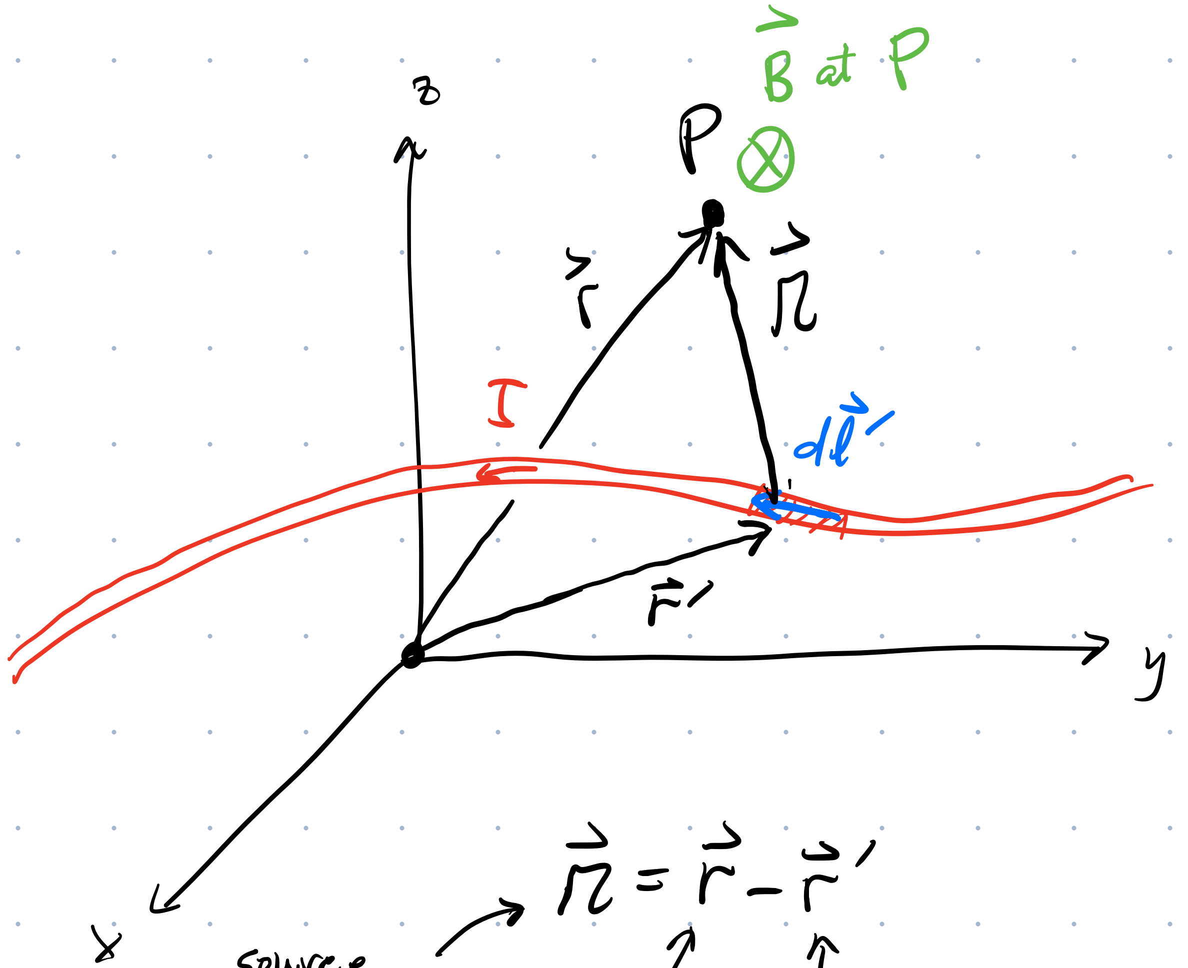
$$\boxed{\vec{\nabla} \cdot \vec{J} = 0}$$

Biot-Savart Law

- describes the magnetic field generated by a steady current. It is an empirical eq'n based on experimental observations.

Find

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, dl'$$



$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

source to P

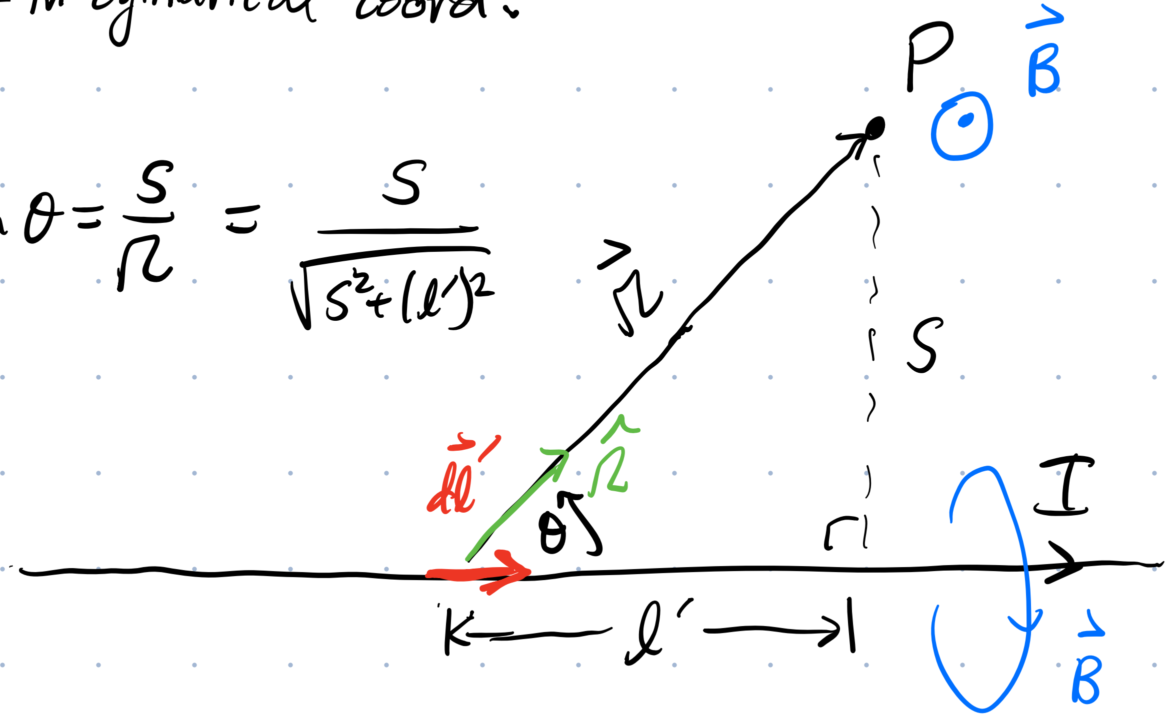
origin to obs. pt. P

origin to source

Eg. $\vec{B}$ due to a long-straight wire.

work in cylindrical coord.

$$\sin\theta = \frac{s}{\mathscr{r}} = \frac{s}{\sqrt{s^2+(l')^2}}$$



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl'$$

$$\vec{I}\times\hat{\mathscr{r}} = I\sin\theta = \frac{I\ s}{\sqrt{s^2+(l')^2}}$$

$$|\vec{B}(\vec{r})| = \frac{\mu_0}{4\pi}\int \frac{I s}{\sqrt{s^2+(l')^2}}\,\frac{1}{(s^2+(l')^2)}\,dl'$$

$$= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{s}{[s^2+(l')^2]^{3/2}} \, dl'$$

long wire.

$$= \frac{\mu_0 I s}{4\pi} \left. \frac{l'}{s^2\sqrt{s^2+(l')^2}} \right|_{l'=-\infty}^{\infty}$$

$$= \frac{\mu_0 I \cancel{s}}{4\pi s^{\cancel{2}}} \left[1-(-1) \right]$$

$$B(\vec{r}) = \frac{\mu_0 I}{2\pi s} \Rightarrow$$ dir'n of $\vec{B}$ in cylindrical coords is $\hat{\phi}$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}}$$