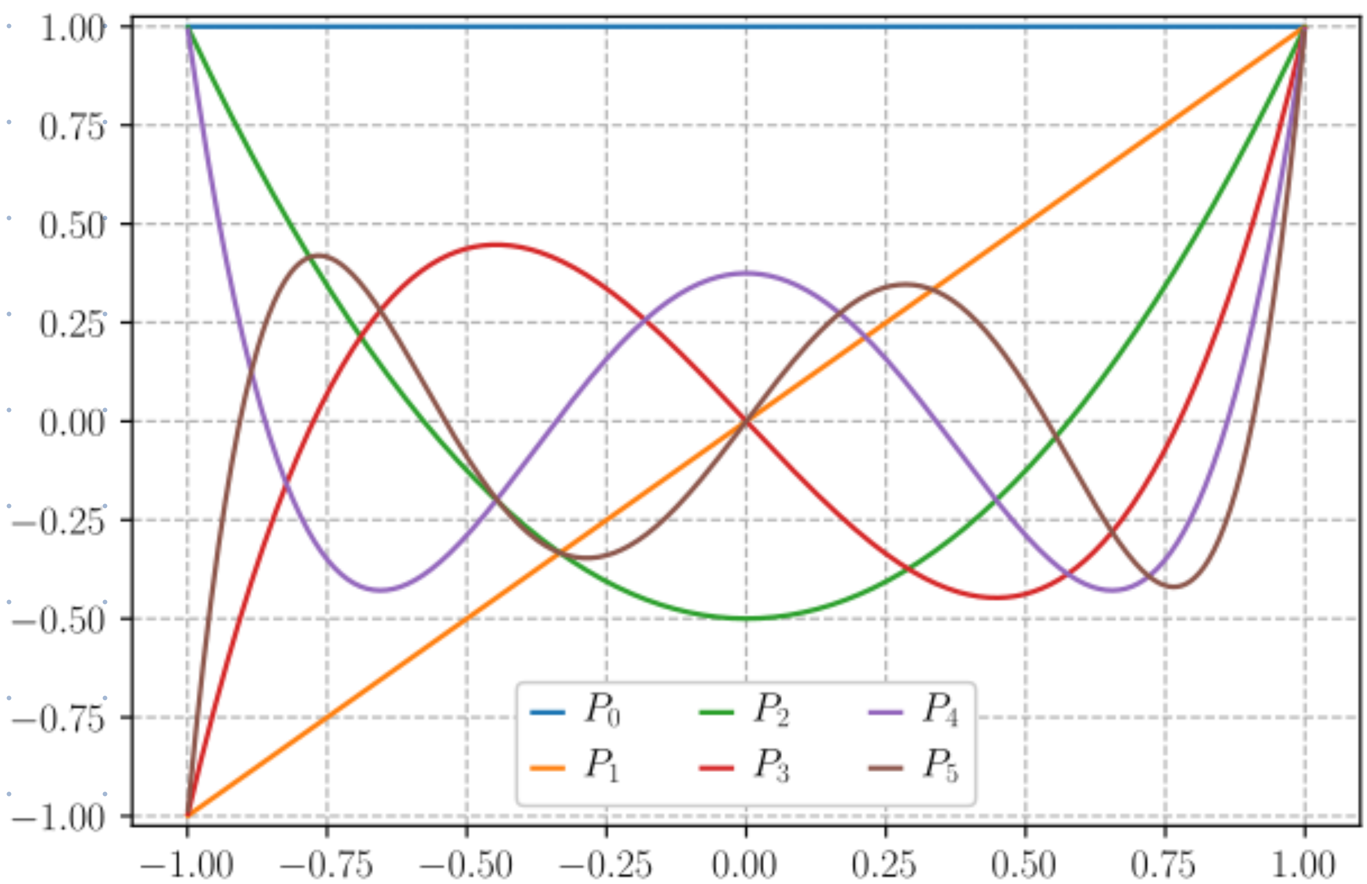
PHYS 301 Oct. 23, 2024

<u>Last Time</u>: Legendre Polynomials &
Inner Products.



$P_\ell(x)$: Polynomials of degree ℓ.

ℓ even → P_ℓ has only even powers of x
ℓ odd → " " " odd " " "

Inner product of two fcns

$$\langle f(x), g(x) \rangle = \int_a^b f(x) g(x)\, dx$$

for $f(x)$, $g(x)$ on $a < x < b$.

If $\langle f(x), g(x) \rangle = 0$, then $f(x)$ & $g(x)$ are said to be orthogonal.

$$\langle P_\ell(x), P_{\ell'}(x) \rangle = 0 \quad \forall \ \ell \neq \ell'$$

$\therefore$ $P_\ell(x)$ forms an orthogonal basis that can be used to construct any function $f(x)$ on $-1 < x < 1$

$$f(x) = \sum_{\ell=0}^{\infty} C_\ell P_\ell(x)$$

where $C_\ell = \langle f(x), P_\ell(x) \rangle = \int_{-1}^{1} f(x) P_\ell(x)\, dx$

Today: - Develop a scheme for constructing an orthogonal set of polynomial fcns

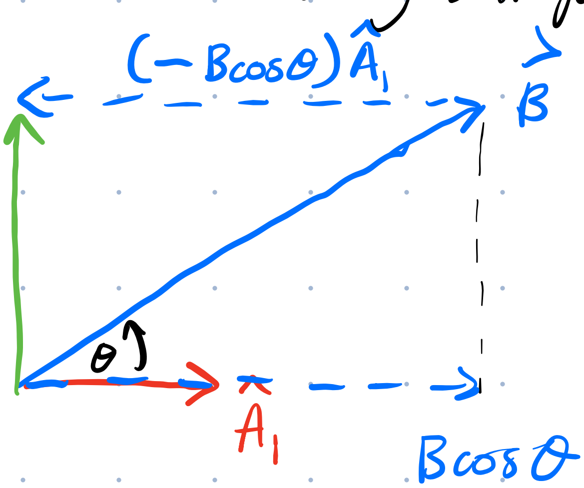
- Show that this set is $P_\ell(x)$ (i.e. the Legendre Polynomials)

If we want to develop a set of orthogonal basis vectors / fcns, what is the procedure?

In general, how can we construct an orthogonal set of vectors?

Suppose $\hat{A}_1$ is going to be one vector in our orthogonal set. How can we take a second vector $\vec{B}$ & use it to construct something orthogonal to $\hat{A}_1$?

$$\vec{B} - B\cos\theta\,\hat{A}_1$$
$$= \vec{B} - (\vec{B}\cdot\hat{A}_1)\hat{A}_1$$



1. Consider $\vec{B}\cdot\hat{A}_1 = |\vec{B}|\underbrace{|\hat{A}_1|}_{A_1=1}\cos\theta = B\cos\theta$

2. Next, consider $\underbrace{\vec{B} - (\vec{B}\cdot\hat{A}_1)\hat{A}_1}_{\text{this vector is } \perp \text{ to } \hat{A}_1}$

$$\left[\vec{B} - (\vec{B}\cdot\hat{A}_1)\hat{A}_1\right]\cdot\hat{A}_1$$

$$= \vec{B}\cdot\hat{A}_1 - (\vec{B}\cdot\hat{A}_1)\underbrace{\hat{A}_1\cdot\hat{A}_1}_{1} = 0. \checkmark$$

Defind $\vec{A}_2 = \vec{B} - (\vec{B}\cdot\hat{A}_1)\hat{A}_1$

we could normalize $\vec{A}_2$ by:

$$\hat{A}_2 = \frac{\vec{A}_2}{|\vec{A}_2|}$$

To construct a 3rd orthogonal vector from $\vec{C}$, it needs to be $\perp$ to both $\hat{A}_1$ & $\hat{A}_2$.

$$\vec{A}_3 = \vec{C} - (\vec{C}\cdot\hat{A}_1)\hat{A}_1 - (\vec{C}\cdot\hat{A}_2)\hat{A}_2$$

$$\vec{A}_3 \perp \hat{A}_1, \hat{A}_2$$

$$\vec{A}_4 = \vec{D} - (\vec{D}\cdot\hat{A}_1)\hat{A}_1 - (\vec{D}\cdot\hat{A}_2)\hat{A}_2 - (\vec{D}\cdot\hat{A}_3)\hat{A}_3$$

$$\vdots$$

This method of constructing an orthogonal set of basis vectors is called the

<u>Gram-Schmidt</u> method.

It can be extend to fcns.

Suppose we start w/ the set fcns

$$G = \{1, x, x^2, x^3, \dots\}$$

use the members of G to find an orthogonal set of fcns.

$$F = \{ f_0, f_1, f_2, \dots \}$$

First, arbitrarily set $f_0 = 1$.

$$f_1 = g_1 - \frac{\langle f_0, g_1 \rangle f_0}{\langle f_0, f_0 \rangle}$$

$$= x - \frac{\left(\cancel{\int_{-1}^{1} 1 \cdot x \, dx} \right) 1}{\int_{-1}^{1} 1 \cdot 1 \, dx} \quad 0$$

$$f_1 = x$$

$$f_2 = g_2 - \frac{\langle f_0, g_2 \rangle f_0}{\langle f_0, f_0 \rangle} - \frac{\langle f_1, g_2 \rangle f_1}{\langle f_1, f_1 \rangle}$$

$$= x^2 - \frac{\left(\int_{-1}^{1} 1 x^2 \, dx\right) 1}{\int_{-1}^{1} dx} - \frac{\cancel{\left(\int_{-1}^{1} x x^2 \, dx\right)}^{\,0} x}{\int_{-1}^{1} x^2 \, dx}$$

$$= x^2 - \frac{\left.\frac{x^3}{3}\right|_{-1}^{1}}{\left. x \right|_{-1}^{1}} = x^2 - \frac{1}{3}$$

$f_0 = 1$

$f_1 = x$

$f_2 = x^2 - \frac{1}{3}$

l	P_l
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$

$$P_\ell = B_\ell f_\ell$$

$\ell = 2$ $\quad \underbrace{\frac{1}{2}(3x^2 - 1)}_{\frac{3}{2}\left(\cancel{x^2 - \frac{1}{3}}\right)} = B_2\left(\cancel{x^2 - \frac{1}{3}}\right)$

$$B_2 = \frac{3}{2}.$$

If you find $f_3 = x^3 - \frac{3}{5}x$

$$f_3 = \frac{5}{2}P_3 \quad \ldots$$

All of the fcns in our orthogonal set F are just scaled versions of the Leg. Poly. $P_\ell(x)$.

One final comment...

A fcn $f(x)$ is said to be normalized if its inner product w/ itself is equal to one.

$$\langle f(x), f(x) \rangle = \int_a^b f(x) f(x)\, dx = 1.$$

The Legendre Polynomials are <u>not</u> normalized

Rather

$$\langle P_\ell(x), P_{\ell'}(x) \rangle = \int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx$$

$$= \delta_{\ell\ell'} \left(\frac{2}{2\ell+1} \right) = \begin{cases} 0 & \ell \neq \ell' \\ \frac{2}{2\ell+1} & \ell = \ell' \end{cases}$$

Griffiths 3.3.2 Separation of Variables in Spherical Coords.

$$\nabla^2 V = \frac{1}{\cancel{r^2}}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\cancel{r^2}\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$$

$$+ \frac{1}{\cancel{r^2}\sin^2\theta}\cancelto{0}{\frac{\partial^2 V}{\partial \phi^2}} = 0$$

We will assume azimuthal symmetry s.t.

$$V = V(r,\theta) \quad \text{and} \quad \frac{\partial V}{\partial \phi} = 0$$

Separation of variables $\Rightarrow$ $V(r,\theta) = R(r)\Theta(\theta)$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}(R\Theta)\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}(R\Theta)\right)$$

$$= 0$$

$$\therefore \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\sin\theta\,\Theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = 0$$

We will set each term equal to a const but w/ opp. sign. It will be convenient to set our constant to be $l(l+1)$

Two eq'n are:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R \quad ①$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = -l(l+1)\Theta \quad ②$$

The general sol'ns to ① are:

$$\boxed{R(r) = Ar^l + \frac{B}{r^{l+1}}}$$

check:

$$\frac{dR}{dr} = l A r^{l-1} - (l+1)\frac{B}{r^{l+2}}$$

$$r^2\frac{dR}{dr} = l A r^{l+1} - (l+1)\frac{B}{r^{l}}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)\underbrace{\left[A r^{l} + \frac{B}{r^{l+1}}\right]}_{R(r)} ✓$$

Next, consider (2)

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta \quad (2)$$

Recall that Legendre Poly. $P_\ell(x)$ were sol'ns to (#)

$$(1-x^2)\frac{d^2y(x)}{dx^2} - 2x\frac{dy(x)}{dx} = -\ell(\ell+1)\,y(x) \quad (\#)$$

Want to show that (2) & (#) are related.

Consider a change of variables in (2).

$$x = \cos\theta \qquad \Theta(\theta) \Rightarrow \Theta(x)$$

$$\sin^2\theta = 1-\cos^2\theta = 1-x^2$$

$$\frac{d}{d\theta} \longrightarrow \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$$

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)$$

$$= -\sin\theta\frac{d}{dx}\left[\sin\theta\,(-\sin\theta)\,\frac{d\Theta}{dx}\right]$$

$$= \sin\theta \frac{d}{dx}\left[\underbrace{\sin^2\theta}_{(1-x^2)} \frac{d\Theta}{dx}\right]$$

$$= \sin\theta \frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right]$$

$$= \sin\theta\left[-2x\frac{d\Theta}{dx} + (1-x^2)\frac{d^2\Theta}{dx^2}\right]$$

$\therefore$ ② becomes.

$$\cancel{\sin\theta}\left[(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx}\right] = -l(l+1)\cancel{\sin\theta}\,\Theta$$

Exactly the form of (#) w/ $x = \cos\theta$.

$\therefore$ sol'ns are the Legendre Polynomials

$$\boxed{\therefore \Theta_l = P_l(\cos\theta)} \qquad l = 0, 1, 2, \ldots, \infty$$

general sol'n for $V(r,\theta)$ is:

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)}$$

Eg. Find $V(r,\theta)$ inside a hollow sphere of radius R. The surface of sphere is at potential $V_0(\theta)$.

Match b.c.s:

(i) $(r=0)$ $\quad V(0,\theta)$ must be finite

(ii) $(r=R)$ $\quad V(R,\theta) = V_0(\theta)$

From (i), $B_l = 0 \ \forall \ l$, otherwise

$V(0,\theta)$ diverges.

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

From (ii)

$$\boxed{V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)}$$

$$= V_0(\theta)$$

Need to select A_l coefficients to match θ-dependence of $V_0(\theta)$.

Already know

$$\int_{-1}^{1} P_l(x) P_{l'}(x) dx$$

$$= \frac{2}{2l+1} \delta_{ll'}$$

$x = \cos\theta$ $\quad$ $x = -1, \ \theta = \pi$

$x = 1, \ \theta = 0$

$$\int_{\pi}^{0} V(R,\theta)\, P_{\ell'}(\cos\theta)\, \underbrace{d(\cos\theta)}_{-\sin\theta\, d\theta}$$

$$= \int_{0}^{\pi} V(R,\theta)\, P_{\ell'}(\cos\theta) \sin\theta\, d\theta$$

$$= \int_{0}^{\pi} \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) \sin\theta\, d\theta$$

$$= \sum_{\ell=0}^{\infty} A_\ell R^\ell \underbrace{\int_{0}^{\pi} P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) \sin\theta\, d\theta}_{\equiv \int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx \,=\, \delta_{\ell\ell'} \frac{2}{2\ell+1}}$$

$$= \sum_{\ell=0}^{\infty} A_\ell R^\ell \delta_{\ell\ell'} \frac{2}{2\ell+1} = A_{\ell'} R^{\ell'} \frac{2}{2\ell'+1}$$

Solving for A_ℓ gives:

$$\boxed{A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi V(R,\theta)\, P_\ell(\cos\theta)\sin\theta\, d\theta}$$

To go further, need to specify $V(R,\theta) = V_0(\theta)$

suppose $V_0(\theta) = k\sin^2\!\left(\frac{\theta}{2}\right)$ k, const.

$$\boxed{\begin{aligned} \sin^2\theta &= \tfrac{1}{2}(1-\cos 2\theta) \\ &\Downarrow \\ \sin^2\tfrac{\theta}{2} &= \tfrac{1}{2}(1-\cos\theta) \end{aligned}}$$

$$= \frac{k}{2}(1-\cos\theta)$$

ℓ	$P_\ell(x)$	$P_\ell(\cos\theta)$
0	1	1
1	x	$\cos\theta$

Notice that $\frac{k}{2}(1-\cos\theta)$

P_0 $\quad$ $P_1(\cos\theta)$

$$V_0(\theta) = \frac{k}{2}(P_0 - P_1)$$

$$\therefore A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi \frac{k}{2}(P_0 - P_1) P_\ell \sin\theta \, d\theta$$

$$= \frac{k}{2}\,\frac{2\ell+1}{2R^\ell}\left[\int_0^\pi P_0 P_\ell \sin\theta \, d\theta - \int_0^\pi P_1 P_\ell \sin\theta \, d\theta\right]$$

b/c P_ℓ's are orthogonal, only

$\ell = 0, 1$ case give non-zero results for A_ℓ.

$\ell = 0$

$$A_0 = \frac{k}{2}\,\frac{1}{2}\left(\frac{2}{2(0)+1}\right) = \boxed{\frac{k}{2}}$$

$$A_1 = -\frac{k}{2}\,\frac{\cancel{3}}{\cancel{2}R}\left(\frac{\cancel{2}}{\cancel{2(1)+1}}\right) = \boxed{-\frac{k}{2R}}$$